The periodic Steiner problem

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Zusammenfassung

Die vorliegende Arbeit befasst sich mit periodischen Netzwerken. Dies sind in den euklidischen Raum $\mathbb{R}^n$ immersierte, zusammenhängende Graphen, die invariant unter Translationen entlang eines Gitters Λ vollen Ranges sind. Wir definieren die Länge L eines periodischen Netzwerks N als die Länge des Quotienten N/Λ und weiterhin sein Volumen V als das n-dimensionale Volumen des flachen Torus $\mathbb{R}^n/\Lambda$. In dieser Arbeit lösen wir das *periodische Steiner-Problem*. Dieses besteht darin, die periodischen Netzwerke zu bestimmen, die den *Längenquotienten* L^n/V minimieren. Dabei sind weder der topologische Graph des Netzwerks noch das Gitter Λ vorgeschrieben.

Motiviert wird das periodische Steiner-Problem durch die Flächentheorie: Minimiert man beispielsweise die Willmore-Energie $\int_\Sigma H^2 \, \mathrm{d}S$ einer dreifach periodischen Fläche Σ_C mit festem Gitter, unter der Bedingung, dass Σ_C ein Gebiet mit Volumen $C > 0$ berandet, dann zeigen numerische Experimente, dass eine kontinuierliche Familie von Flächen Σ_c für $c \in (0, C)$ existiert. Für $c \to 0$ degeneriert die Fläche in ein periodisches Netzwerk N. Für $c \approx 0$ besteht die Fläche Σ_c aus dünnen zylindrischen Röhren, die als Tuben um das Netzwerk N angesehen werden können. Im ersten Kapitel zeigen wir durch Berechnung der zweiten Variation der Willmore-Energie, dass Zylinder tatsächlich stabile Willmore-Flächen unter Volumen-Nebenbedingung sind.

Im zweiten Kapitel widmen wir uns dem endlichen Steiner-Problem mit fester Topologie: Zu einer vorgegebenen Menge von Punkten wird ein immersierter Graph kürzester Länge gesucht, der diese Punkte miteinander verbindet. Der Minimierer wird in einer vorgeschriebenen Klasse topologischer Graphen gesucht. Die Existenz eines solchen Minimierers folgt mit einem einfachen Kompaktheitsargument; wir untersuchen die Eindeutigkeit

des Minimierers. Ivanov und Tuzhilin [IT94] zeigen, dass ein eingebetteter Minimierer stets eindeutig ist. Wir verallgemeinern dieses Resultat auf allgemeine, nicht-eingebettete Graphen und finden eine geometrische Bedingung, die äquivalent zur Eindeutigkeit ist.

Der Hauptteil der Dissertation ist der dritte Teil. Zunächst beweisen wir, dass Minima des Längenquotienten L^n/V in beliebiger Dimension n genau von denjenigen Netzwerken angenommen werden, die in jedem Knoten die *Steiner-Bedingung* erfüllen. Das bedeutet, dass jeder Knoten des Netzwerks vom Grad 3 ist und sich die drei inzidenten Kanten an jedem Knoten im 120°-Winkel treffen. Solche Netzwerke bezeichnen wir als *periodische Steiner-Netzwerke*. Wir zeigen, dass der Quotientengraph eines Minimierers in Dimension n genau $2n-2$ Knoten hat und beschränken damit die Anzahl der möglichen topologischen Graphen eines Minimierers. Anschließend bestimmen wir den Minimierer in Dimension drei: Unter Verwendung der Steiner-Bedingung zeigen wir für jedes periodische Steiner-Netzwerk, dass der Längenquotient lediglich durch die Kantenlängen des Netzwerks (und einen weiteren Parameter in einem Fall) bestimmt ist, und berechnen den Längenquotienten in diesen Fällen explizit. So identifizieren wir den eindeutigen Minimierer in $\mathbb{R}^3$ als das so genannte srs-Netzwerk (Fig. 3).

Im vierten Kapitel betrachten wir periodische Netzwerke, deren Knoten einen vorgeschriebenen Grad $d \geq 4$ haben, und bestimmen ebenfalls Minimierer des Längenquotienten L^n/V. Auch hier sind wir speziell an Dimension $n = 3$ interessiert, denn die minimierenden Netzwerke reproduzieren die Topologien bekannter periodischer Minimalflächen. Da sich unsere Argumente leicht auf beliebige Dimensionen verallgemeinern lassen, werden zunächst die Minimierer für $d = n + 1$ und $d = 2n$ bestimmt. Im Fall $n = 3$ ergeben sich so für $d = 4$ das bekannte Diamant-Netzwerk und für $d = 6$ das primitive kubische Netzwerk. Um ein vollständiges Bild in Dimension $n = 3$ zu erhalten, werden schließlich die Minimierer vom Grad $d = 5$ durch Betrachtung von expliziten Einbettungen 5-regulärer Graphen berechnet. Diese treten in natürlichen Systemen seltener auf und stellen sich auch als deutlich länger heraus (Tab. 1).

Contents

Introduction

Minimal surfaces are surfaces that are locally area-minimizing. These surfaces are used to model chemical and biological systems, such as photonic crystals [MS08]. Particularly of interest are minimal surfaces that are triply periodic in the sense of being invariant under a 3-dimensional lattice of translations. Such embedded triply periodic minimal surfaces were first constructed by Schwarz and his students. In 1970, Alan Schoen [Sch70] used *skeletal graphs* in order to suggest further candidates for triply periodic minimal surfaces. The skeletal graph of a surface can be thought of as the end result of shrinking the surface along the direction of its normal vectors until the surface degenerates into a graph (cf. Figure 1). Many functionals such as area and Willmore energy of a surface relate to the length of its skeletal graph.

Conversely, one could attempt to construct periodic minimal surfaces by considering periodic graphs, which we call *networks*, and identify a related minimal surface admitting the network as its skeletal graph. For instance, Schoen discovered the gyroid minimal surface in terms of a length minimizing network [Sch70]; rigorous existence proofs were later obtained by Karcher [Kar89] (see also [GBW96]).

The notion of a skeletal graph of a surface, however, does not have a precise mathematical definition. An attempt to define graphs for arbitrary minimal or constant mean curvature Alexandrov embedded surfaces (not necessarily periodic) is due to Kusner [Kus91]: He defines straight lines in terms of loop integrals which are well-defined on the first homology of the surface. However, only in symmetric cases will these lines meet at vertices and thereby define edges of a network.

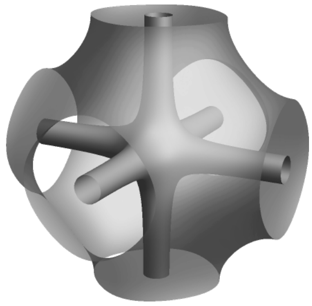

Figure 1: A fundamental domain of the triply periodic Schwarz-P surface, and its skeletal graph.

There is another possible approach to produce the skeletal graph of a surface, at least for very symmetric cases [GB97]. For fixed lattice Λ and constant $C > 0$, suppose there is a periodic embedded surface $\Sigma \subset \mathbb{R}^3$ minimizing the Willmore energy $\int_\Sigma H^2 \,\mathrm{d}S$ in $\mathbb{R}^3/\Lambda$, under the constraint that Σ bounds a component Ω with enclosed volume C. Experiments with Brakke's Surface Evolver indicate that a continuous deformation family Σ_c exists for $c \in (0, C]$, with volume c of the component Ω_c deforming Ω. In many cases a network arises as the singular limit $\lim_{c\to 0} \Sigma_c$ with respect to Hausdorff distance; geometrically, the surfaces Σ_c can be described as thin cylindrical tubes around the skeletal graph.

In Chapter 1 we verify these experiments. We offer a rigorous proof that cylinders are stable Willmore surfaces under a volume constraint (Theorem 1.3). Since the Willmore energy of a cylinderical tube with prescribed volume is determined by its length, this supports the experimental finding that embedded periodic surfaces minimizing the Willmore energy degenerate in the limiting case into length minimizing networks.

These networks are related to the Steiner tree problem. In Chapter 2 we consider the Steiner tree problem with boundary vertices in Euclidean space: Fix a topological graph and determine a length minimizing immersion into Euclidean space connecting a prescribed set of points. Though the existence of a minimizer is immediate by a compactness argument, proving its uniqueness, on the other hand, is more involved. In their book [IT94], Ivanov and Tuzhilin consider general ambient spaces and show uniqueness of minimizers in Euclidean space under additional assumptions. In particular, they prove uniqueness of a minimizer under the assumption that the minimizer is embedded. Our main result of the chapter (Theorem 2.13) generalizes this result to possibly degenerate graphs, i.e., graphs with vanishing edge lengths. By introducing an equivalence relation on the set of vertices we characterize uniqueness of minimizers for degenerate graphs, thereby offering a comprehensive picture of the Steiner tree problem with fixed topology.

We conclude Chapter 2 by considering the classical Steiner problem with free topology: Given a finite set of points, the Steiner problem is to find a tree of minimal length connecting them. Trees minimizing length usually have further vertices which necessarily are of degree 3, where the incident edges are coplanar and meet at 120°-angles. This is valid for any dimension, and we call it the *Steiner condition*. While this is well-known for graphs with boundary vertices the techniques will apply to the periodic cases considered later in the thesis.

The results of Chapters 3 and 4 are in form of publications (see [AGB17] and [AGB18]). They are joint work with my advisor, Karsten Große-Brauckmann.

In Chapter 3 we consider networks minimizing length in the sense we describe now and which is illustrated by Figure 2 for the two-dimensional case. Let Λ be the lattice of a triply periodic network $N \subset \mathbb{R}^3$. Then the fundamental domain $\mathbb{R}^3/\Lambda$ is a flat 3-torus with volume V, and the network quotient N/Λ has a length L. Since scaling can reduce the length of N, a well-posed variational problem is: *Minimize the network length L under the constraint $V = 1$.* Equivalently, one can minimize the scale-invariant quotient L^3/V. The first part of our main result (see below) determines the topology of the length minimizer in arbitrary dimensions. In particular, we show that minimizers are *Steiner networks*, that is, the Steiner condition is met at all vertices. The second part of the Theorem identifies the unique minimizer in dimension 3 as the srs network.

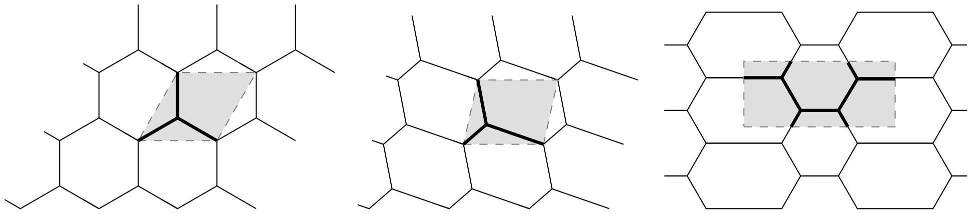

Figure 2: Doubly periodic Steiner networks and their fundamental domains. The underlying abstract graph of the first two networks is the dipole graph of order 3. The first network, with the hexagonal lattice, minimizes length for given area of its fundamental domain. The network on the right has the quotient $D_1 \square D_2$.

The srs network, shown in Figure 3, is highly symmetric, with symmetry group $I4_132$. It is the skeletal graph of Schoen's gyroid minimal surface. Its quotient under the body-centred cubic lattice is the complete graph on 4 vertices K_4. However, Steiner networks with 4 vertices in the quotient exist for arbitrary lattices.

Many familiar networks have vertices with a degree higher than 3, such as the diamond network with degree 4. In fact, for a natural system in Euclidean space, material reasons may be present which prescribe a degree $d > 3$ at the vertices. For simplicity, we consider only the so-called d-regular case that d agrees at all vertices. Then it seems natural to ask: *What are the triply periodic networks in $\mathbb{R}^3$ minimizing L^3/V among networks with a prescribed degree $d \geq 4$?* We also ask: *How much larger is L^3/V for $d \geq 4$ compared with the case $d = 3$?* In Chapter 4 we address these questions for networks whose quotient N/Λ has the minimal number of vertices (Theorems 4.13, 4.15 and 4.16), a case we call *irreducible*.

For Euclidean space $\mathbb{R}^3$, the results of Chapters 3 and 4 give a complete picture. Our determination of the minimal length quotient seems nicely consistent with the occurence of the networks in natural systems, although the reasons leading to the networks in nature are certainly more complex. Indeed, as Table 1 indicates, the length quotient $L/V^{1/3}$ for the frequently encountered diamond network is only by 3% larger than for the optimal Steiner network srs. The two families ths and cds admit deformations into

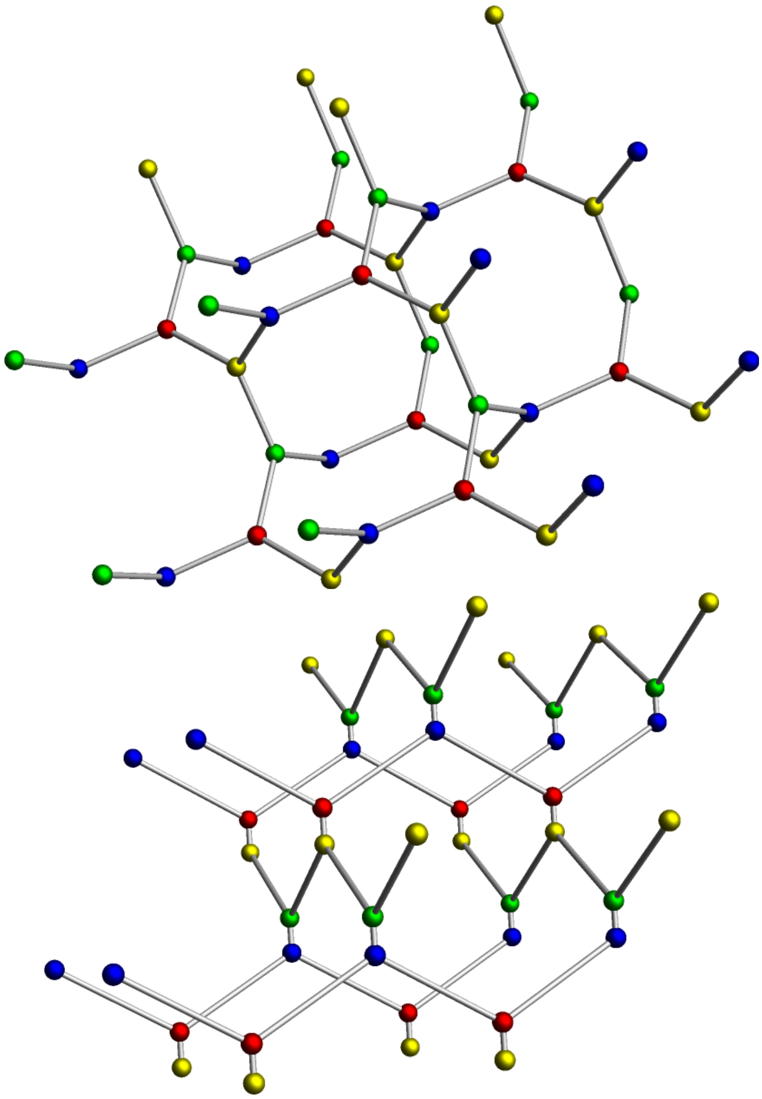

Figure 3: We identify the srs network (top) as the length minimizer in the class of all triply periodic networks. As indicated by the colouring, the quotient has four vertices and is the graph K_4. Triply periodic Steiner networks on four vertices can also have the graph $D_1 \square D_2$ as a quotient; a minimizing ths network is depicted on the bottom. Observe that the long edges define zigzag curves which are contained in perpendicular planes. The short edges are contained in lines of intersection of these planes.

degree	$d = 3$		$d = 4$		$d = 5$		$d = 6$
#vertices/#edges	4 / 6		2 / 4		2 / 5		1 / 3
graph	K_4	$D_1 \square D_2$	D_4	$D_{1,2}$	$D_{1,3}$	D_5	B_3
minimizer	srs	ths	dia	cds	bnn	sqp	pcu
	unique	family	unique	family	unique	unique	unique
related surface	gyroid	—	D	CLP	H'-T	—	P
$L/V^{1/3}$	≈ 2.67	≈ 2.73	≈ 2.75	3	≈ 3.6	≈ 3.7	3
	100%	102.0%	102.9%	112.3%	134.8%	135.2%	112.3%

Table 1: Minimizing triply periodic networks with prescribed degree 3 to 6 and their length quotients; all networks with the least possible number of vertices in the quotient are studied. See text for acronyms of minimizers and surfaces.

networks of smaller length, and so are less likely to occur. Thus the next best network is pcu with degree $d = 6$, and a quotient by 12% larger than srs. There is a significant gap to the networks with $d = 5$, which seem of minor physical importance, as their quotient $L/V^{1/3}$ is by 35% larger compared to srs. Finally, no network with degree $d \geq 7$ can have a smaller quotient.

The notation for the abstract graphs is explained in Chapter 4. The acronyms for the minimizing networks quoted in the fourth line of Table 1 are chosen to conform to the notation used by crystallographers, see [HOP08] and also rcsr.net. We should note, however, that the lengths of our minimizers differ in some cases from the crystallographic standard representations, where edge lengths are chosen to coincide whenever possible. For completeness, let us explain the acronyms. In many cases they refer to a chemical compound: For the Steiner networks, srs stands for $SrSi_2$ and ths for $ThSi_2$. The diamond form of carbon explains dia, and cds stands for $CdSO_4$, while bnn denotes boron nitride nanotubes. Some other networks are named according to their lattice or geometry: pcu denotes the primitive cubic unit, sqp denotes a network composed of square pyramids; in the two-dimensional case, sql relates to the square lattice and hcb to the hexagonal or honeycomb network.

While originally our interest was solely in the case of Euclidean space $n =$

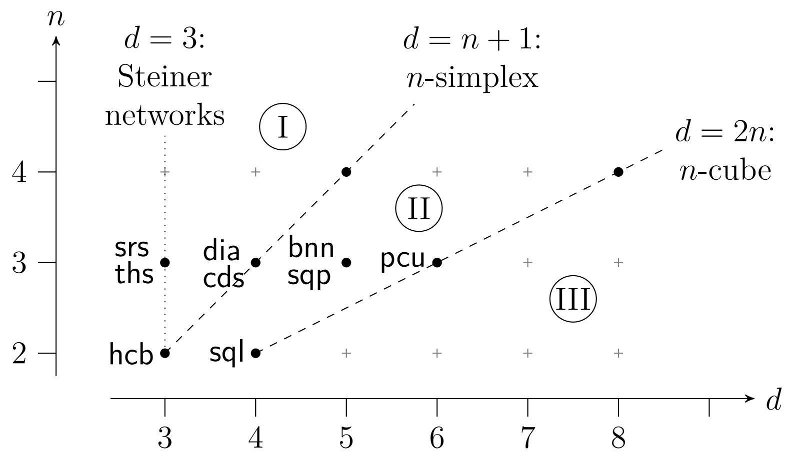

Figure 4: The minimizers determined in this thesis are indicated with black dots; for dimension $n = 3$ these are described in Table 1. The two dashed lines denote results for families of minimizers. For the enclosed regions II and III we have estimates of the length quotient.

3, we have come to study higher dimensions as well. One reason is that the case of general dimension indicates which features are open to a systematic study, and which others seem only accessible to a case-by-case study. Another reason is that some of our techniques are natural to state in arbitrary dimension n.

Figure 4 shows how the results for general n relate to the ones for $n = 3$. The minimizing simplicial networks with $d = n + 1$ generalize the diamond or the hexagonal planar network to arbitrary dimension, and the primitive cubical networks minimizing for $d = 2n$ generalize the primitive network in 3-space or the planar square lattice.

For the region with $d \leq n$ marked with I in Figure 4, the network quotients must contain more than 2 vertices. As shown in Chapter 3, for the Steiner case $d = 3$ the quotient graph has at least $2n - 2$ vertices. It is known that the number of topologically different graphs with $2n - 2$ vertices rapidly increases with n. Thus we do not expect a good systematic theory for the case $d \leq n$.

While we do not offer a characterization of the minimizers with d between $n + 2$ and $2n - 1$, corresponding to the region II of the figure, we show that for each n the corresponding minimizers have a length larger than the simplicial networks generalizing the diamond. Similar remarks apply to region III with $d > 2n + 1$: Here the primitive cubic network gives rise to a lower estimate. That is, in regions II and III of the figure, the length quotient is estimated strictly by the minimizers represented by the dashed lines to their left.

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1 Periodic Willmore surfaces

The present chapter identifies cylinders as stable Willmore surfaces under a volume constraint. We start by giving a short introduction to the Willmore energy. Let Σ be a compact oriented two-dimensional surface and $f\colon \Sigma \to E^3$ be an immersion into a flat three-dimensional space E^3. The Willmore energy is defined as

$$W(f) := \int_\Sigma H^2 \,\mathrm{d}S\,,$$

where H is the mean curvature of f and $\mathrm{d}S$ is the volume element of Σ.

A remarkable property of the Willmore energy is its invariance under conformal deformations of the ambient space. Clearly, the Willmore energy is invariant under similiarities of E^3. Proving the invariance under inversions, however, is more involved. A direct computation of the Willmore energy $W(g \circ f)$ for inversions g can be found in Willmore's book [Wil96, Theorem 7.3.1]. A geometric reasoning for this property was suggested by Eschenburg [Esc13]: Inversions send the two principal curvature spheres S_i of f with radii $1/\kappa_i$ onto principal curvature spheres $g(S_i)$ of $g \circ f$ with radii $1/\kappa_i'$ such that

$$|\kappa_1 - \kappa_2| = |\kappa_1' - \kappa_2'|\,.$$

Rewriting the Willmore energy of f as

$$W(f) = \frac{1}{4}\int_\Sigma (\kappa_1+\kappa_2)^2 \,\mathrm{d}S = \frac{1}{4}\int_\Sigma (\kappa_1-\kappa_2)^2 \,\mathrm{d}S + \int_\Sigma \kappa_1\kappa_2 \,\mathrm{d}S$$

and applying the Gauß-Bonnet theorem then yields $W(f) = W(g \circ f)$.

1.1 Euler-Lagrange equation for the Willmore energy

Before we discuss the Willmore energy with a volume constraint imposed, let us review the Euler-Lagrange equation satisfied by critical points of the Willmore energy. Here we follow the derivation Willmore gave in his book [Wil96].

Consider an immersion $f\colon \Sigma \to E^3$ with a choice of normal ν. Here, Σ is an oriented closed surface. Let $u\colon \Sigma \to \mathbb{R}$ be a real-valued, compactly supported smooth function and choose $\varepsilon > 0$ sufficiently small such that $f^t := f + tu\nu$ is an immersion for all $|t| < \varepsilon$. Then, the Willmore energy of the perturbed surface f^t is well-defined and

$$(-\varepsilon, \varepsilon) \to \mathbb{R}\,, \qquad t \mapsto W(f^t)$$

is smooth at $t = 0$. In this section we prove the expansion for the Willmore energy,

$$W(f^t) = W(f) + t\int_\Sigma u\Big(\Delta_g H + 2H(H^2 - K)\Big)\,\mathrm{d}S + O(t^2)\,, \tag{1.1}$$

where $\Delta_g H$ is the Laplace-Beltrami operator applied to the mean curvature H and K is the Gauß curvature. Using the fundamental lemma of calculus of variations (see [GF12]) we then obtain the Euler-Lagrange equation

$$\Delta_g H + 2H(H^2 - K) = 0\,.$$

Remark 1.1. The Euler-Lagrange equation characterizes surfaces critical for the Willmore energy. In order to compute the Euler-Lagrange equation, it suffices to consider variations of the form $f^t := f + tu\nu$, i.e., without second or higher order terms. Investigating the stability of Willmore surfaces, however, requires taking into account effects of second and higher order.

A particular choice of normal ν is $f_1 \times f_2/|f_1 \times f_2|$, where we use the notation $f_i := \partial f/\partial x_i$ for $i = 1, 2$. Denote by g the first fundamental form and by b the second fundamental form of Σ. The coefficients of these matrices are given by $g_{ij} = \langle f_i, f_j \rangle$ and

$$b_{ij} = -\langle f_i, \nu_j \rangle = \langle f_{ij}, \nu \rangle = b_{ji}\,, \quad \nu_j := \frac{\partial \nu}{\partial x_j}\,, \tag{1.2}$$

where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ denotes the second partial derivative of f. If we denote the inverse matrix of (g_{ij}) by (g^{ij}) then the mean curvature is given by

$$H = \frac{1}{2} \sum_{i,j} g^{ij} b_{ij} = \frac{1}{2 \det g} (g_{11} b_{22} + g_{22} b_{11} - 2 g_{12} b_{12})\,. \tag{1.3}$$

Our goal is to write the Willmore energy for the surface f^t in terms of the surface f. So we need to compute the first order terms of H^t and $\sqrt{\det g^t}$. We have

$$\begin{aligned} g^t_{ij} &= \langle f_i + t u_i \nu + t u \nu_i, f_j + t u_j \nu + t u \nu_j \rangle \\ &= \langle f_i, f_j \rangle + t u \langle f_i, \nu_j \rangle + t^2 u_i u_j \langle \nu, \nu \rangle + t u \langle \nu_i, f_j \rangle + t^2 u^2 \langle \nu_i, \nu_j \rangle \\ &= g_{ij} - 2 t u b_{ij} + O(t^2) \end{aligned}$$

for the first fundamental form. In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t} g^t_{ij} \Big|_{t=0} = -2 u b_{ij}\,. \tag{1.4}$$

This gives the expansion

$$\begin{aligned} \det g^t &= g^t_{11} g^t_{22} - (g^t_{12})^2 \\ &= g_{11} g_{22} - (g_{12})^2 - 2tu(g_{11} b_{22} + g_{22} b_{11} - 2 g_{12} b_{12}) + O(t^2)\,. \end{aligned}$$

Using (1.3) we obtain

$$\det g^t = \det g - 4 t u H \det g + O(t^2)\,. \tag{1.5}$$

We compute the root of the determinant $\sqrt{\det g^t}$ with Taylor's formula

$$\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)\,.$$

Thus (1.5) gives

$$\sqrt{\det g^t} = \sqrt{\det g}\,(1 - 2tuH) + O(t^2)\,. \tag{1.6}$$

With the product rule we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H^t = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i,j}(g^t)^{ij}b^t_{ij} = \frac{1}{2}\sum_{i,j}\Big(\frac{\mathrm{d}}{\mathrm{d}t}(g^t)^{ij}b_{ij} + g^{ij}\frac{\mathrm{d}}{\mathrm{d}t}b^t_{ij}\Big)\,. \tag{1.7}$$

We compute $\frac{\mathrm{d}}{\mathrm{d}t}(g^{ij})^t$ and $\frac{\mathrm{d}}{\mathrm{d}t}b^t_{ij}$ at $t=0$ separately. The relation $\sum_j g_{ij}g^{jk} = \delta^k_i$ gives

$$\sum_j \frac{\mathrm{d}}{\mathrm{d}t}g^t_{ij}g^{jk} + \sum_j g_{ij}\frac{\mathrm{d}}{\mathrm{d}t}(g^t)^{jk} = 0\,.$$

Using (1.4) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(g^t)^{ij}\bigg|_{t=0} = 2u\sum_{k,l} g^{jk}g^{il}b_{kl}\,. \tag{1.8}$$

Differentiating $\langle \nu^t, f^t_i\rangle = 0$ and applying the product rule yields

$$\Big\langle\frac{\mathrm{d}}{\mathrm{d}t}\nu^t, f_i\Big\rangle\bigg|_{t=0} = -\,\Big\langle\nu, \frac{\mathrm{d}}{\mathrm{d}t}f^t\Big\rangle\bigg|_{t=0} = -\langle\nu, u_i\nu + u\nu_i\rangle = -u_i\,. \tag{1.9}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\nu^t$ is orthogonal to ν at $t=0$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\nu^t\bigg|_{t=0} = \sum_j a^j f_j$$

with some coefficients $a^j \in \mathbb{R}$. Using (1.9) they can be computed as

$a^j = -\sum_i g^{ij} u_i$. We now invoke the Gauß equation

$$f_{ij} = \sum_k \Gamma^k_{ij} f_k + b_{ij}\nu\,,$$

where

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{lk}\Big(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ik} - \frac{\partial}{\partial x_k} g_{ij}\Big)$$

denotes the Christoffel symbols. From the Gauß equation we obtain

$$\begin{aligned}\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\nu^t, f_{ij}\Big\rangle\Big|_{t=0} &= -\sum_{p,q,k} \langle g^{pq} u_q f_p, \Gamma^k_{ij} f_k\rangle \\ &= -\sum_{p,q,k} g^{pq} u_q \Gamma^k_{ij} g_{pk} = -\sum_{q,k} \delta^q_k u_q \Gamma^k_{ij} = -\sum_k u_k \Gamma^k_{ij}\,.\end{aligned} \tag{1.10}$$

The second partial derivatives of f^t,

$$f^t_{ij} = f_{ij} + t(u_{ij}\nu + u_i\nu_j + u_j\nu_i + u\nu_{ij})\,,$$

give

$$\Big\langle \nu, \frac{\mathrm{d}}{\mathrm{d}t} f^t_{ij}\Big\rangle\Big|_{t=0} = u_{ij} + u\langle \nu, \nu_{ij}\rangle = u_{ij} - u\langle \nu_i, \nu_j\rangle\,. \tag{1.11}$$

We insert the Weingarten equations,

$$\nu_i = -\sum_{j,k} g^{jk} b_{ik} f_j\,, \tag{1.12}$$

into (1.11) to obtain

$$\begin{aligned}\Big\langle \nu, \frac{\mathrm{d}}{\mathrm{d}t} f^t_{ij}\Big\rangle\Big|_{t=0} &= u_{ij} - u\sum_{k,l,p,q} \langle g^{pk} b_{ik} f_p, g^{ql} b_{jl} f_q\rangle \\ &= u_{ij} - u\sum_{k,l} b_{ik} g^{kl} b_{lj}\,.\end{aligned} \tag{1.13}$$

Combining (1.13) and (1.10) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} b^t_{ij}\Big|_{t=0} &= \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\nu^t, f_{ij}\Big\rangle\Big|_{t=0} + \Big\langle \nu, \frac{\mathrm{d}}{\mathrm{d}t} f^t_{ij}\Big\rangle\Big|_{t=0} \\ &= -\sum_k \Gamma^k_{ij} u_k + u_{ij} - u \sum_{k,l} b_{ik} g^{kl} b_{lj}\end{aligned} \tag{1.14}$$

for the second fundamental form. Inserting (1.8) and (1.14) into (1.7) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H^t\Big|_{t=0} &= u \sum_{i,j,k,l} g^{jk} g^{il} b_{kl} b_{ij} + \frac{1}{2}\sum_{i,j} g^{ij}\Big(- \sum_k \Gamma^k_{ij} u_k + u_{ij} - u \sum_{k,l} b_{ik} g^{kl} b_{lj}\Big) \\ &= \frac{1}{2}\sum_{i,j} g^{ij}(-\sum_k u_k \Gamma^k_{ij} + u_{ij}) + \frac{1}{2} u \sum_{i,j,k,l} g^{ij} b_{ik} g^{kl} b_{lj} \\ &= \frac{1}{2}\Big(\Delta_g u + u \sum_{i,j,k,l} g^{ij} b_{ik} g^{kl} b_{lj}\Big) \\ &= \frac{1}{2}\Big(\Delta_g u + u \,\mathrm{trace}\Big((g^{-1}b)^2\Big)\Big) .\end{aligned}$$

Here, $\Delta_g u$ denotes the Laplace-Beltrami operator applied to the variation u. The shape operator $g^{-1}b$ has as eigenvalues the principal curvatures κ_1, κ_2. The trace of its square is hence given by

$$\mathrm{trace}\Big((g^{-1}b)^2\Big) = \kappa_1^2 + \kappa_2^2 = 4H^2 - 2K .$$

Thus we get

$$2\,\frac{\mathrm{d}}{\mathrm{d}t} H^t\Big|_{t=0} = \Delta_g u + u(4H^2 - 2K) .$$

With (1.6) we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma (H^t)^2 \,\mathrm{d}S^t\Big|_{t=0} &= \int_\Sigma H\Delta_g u + uH(4H^2 - 2K) - 2uH^3 \,\mathrm{d}S \\ &= \int_\Sigma H\Delta_g u + 2uH(H^2 - K)\,\mathrm{d}S .\end{aligned}$$

Since u is compactly supported it follows from Green's Theorem

$$\int_\Sigma H\,\Delta_g u\,\mathrm{d}S = \int_\Sigma u\,\Delta_g H\,\mathrm{d}S\,.$$

Thus we get the Willmore energy expansion (1.1), as desired.

1.2 Willmore surfaces under a volume constraint

Consider a cylinder C_r^ℓ with radius $r > 0$ and length $\ell > 0$. Since C_r^ℓ has constant mean curvature $H = \frac{1}{2r}$ its Willmore energy is given by

$$W(C_r^\ell) = \mathrm{area}(C_r^\ell)\,H^2 = 2\pi r\ell\,\frac{1}{4r^2} = \frac{\pi\ell}{2r}\,. \tag{1.15}$$

Hence, leaving the length ℓ unchanged, its Willmore energy can be reduced by increasing the radius r of C_r^ℓ. Indeed, $W(C_r^\ell) \to 0$ as $r \to \infty$. Note that increasing the radius r increases the volume enclosed by the cylinder. We want to prove that if we constrain the volume, the cylinder becomes a stable critical point for the Willmore energy. This is verified in numerical experiments and will be shown in Theorem 1.3.

To make this assertion precise, we consider the cylinder C_r^ℓ as a simply periodic surface with period ℓ. Hence we take the torus

$$\Sigma := \mathbb{R}^2/\ell\,\mathbb{Z}\times 2\pi\mathbb{Z}$$

as the domain of our parametrisation and let

$$f\colon \Sigma \to \mathbb{R}^3/\Big(\ell\,\mathbb{Z},0,0\Big)\,, \quad (x,y)\mapsto \Big(x, r\cos y, r\sin y\Big)\,. \tag{1.16}$$

We choose the inner normal to the cylinder

$$\nu\colon \Sigma\to\mathbb{S}^2\,, \qquad \nu(x,y) := (0,-\cos y,-\sin y)\,.$$

Let

$$f^t := f + tu\nu + t^2 w\nu + O(t^3)$$

be a periodic normal variation of f with real-valued functions $u, w\colon \Sigma \to \mathbb{R}$. Here, we consider variations including second order terms. This is necessary in order to prove that cylinders are stable Willmore surfaces under a volume constraint: We will see in the expansion formulas below for the enclosed volume and the Willmore energy, (1.17) and (1.19), that the second variation of both functionals depend on the second order terms of the variation $f^t = f + tu + t^2 w + O(t^3)$.

Using Green's theorem, the volume enclosed by the periodic variation f^t has the expansion

$$\begin{aligned} V(f^t) &= \frac{1}{2}\int_{-\pi}^{\pi}\int_0^{\ell} \left(r - tu - t^2 w\right)^2 \,\mathrm{d}x\,\mathrm{d}y + O(t^3) \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\int_0^{\ell} r^2 + t^2u^2 - 2tru - 2rt^2 w \,\mathrm{d}x\,\mathrm{d}y + O(t^3) \\ &= \pi\ell r^2 - t\,r\int_{-\pi}^{\pi}\int_0^{\ell} u \,\mathrm{d}x\,\mathrm{d}y + \frac{t^2}{2}\int_{-\pi}^{\pi}\int_0^{\ell} u^2 - 2rw \,\mathrm{d}x\,\mathrm{d}y + O(t^3)\,. \end{aligned} \tag{1.17}$$

We call a variation $f^t = f + tu\nu + t^2 w\nu + O(t^3)$ *volume preserving (up to second order)* if

$$\frac{\mathrm{d}}{\mathrm{d}t}V(f^t)\bigg|_{t=0} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}V(f^t)\bigg|_{t=0} = 0\,,$$

that is, u, w satisfy the two equations

$$\int_{-\pi}^{\pi}\int_0^{\ell} u \,\mathrm{d}x\,\mathrm{d}y = \int_{-\pi}^{\pi}\int_0^{\ell} u^2 - 2rw \,\mathrm{d}x\,\mathrm{d}y = 0\,. \tag{1.18}$$

1.3 Stability of cylinders

Out main result is a stability result. We show that for the cylinder any volume preserving variation f^t is critical for the Willmore energy $W(f^t)$ and that the second variation of $W(f^t)$ is non-negative.

First, we compute for the Willmore energy an expansion for variations f^t of the cylinder.

Lemma 1.2. *Let $f^t = f + tu\nu + t^2 w\nu + O(t^3)$ be a normal variation of the cylinder C_r^ℓ. Then the expansion of the Willmore energy is given by*

$$\begin{aligned} W(f^t) &= \frac{\pi\ell}{2r} + \frac{t}{4r^2}\int_{-\pi}^{\pi}\int_0^\ell u \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{t^2}{8r^3}\int_{-\pi}^{\pi}\int_0^\ell 2u^2 - r^2u_1^2 - 5u_2^2 + 4r^2u_{12}^2 + 2r^4u_{11}^2 + 2u_{22}^2 + 2rw \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + O(t^3)\,. \end{aligned} \tag{1.19}$$

Proof. The surface $f^t = f + tu\nu + t^2w\nu + O(t^3)$ is parametrised by

$$f^t(x,y) = \Big(x, (r - tu - t^2w)\cos y, (r - tu - t^2w)\sin y\Big) + O(t^3)\,.$$

The Willmore energy of f^t can be written as

$$\begin{aligned} W(f^t) &= \int_\Sigma (H^t)^2 \,\mathrm{d}S^t \\ &= \frac{1}{4}\int_{-\pi}^{\pi}\int_0^\ell \frac{1}{(\det g^t)^2}(g_{11}^t b_{22}^t + g_{22}^t b_{11}^t - 2g_{12}^t b_{12}^t)^2\sqrt{\det g^t}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{4}\int_{-\pi}^{\pi}\int_0^\ell (\det g^t)^{-5/2}\Big(\big(g_{11}^t b_{22}^t + g_{22}^t b_{11}^t - 2g_{12}^t b_{12}^t\big)\sqrt{\det g^t}\Big)^2 \,\mathrm{d}x\,\mathrm{d}y\,. \end{aligned} \tag{1.20}$$

We set $X^t := g_{11}^t b_{22}^t\sqrt{\det g^t} + g_{22}^t b_{11}^t\sqrt{\det g^t} - 2g_{12}^t b_{12}^t\sqrt{\det g^t}$ and compute $(\det g^t)^{-5/2}$ and $(X^t)^2$ separately. The first partial derivatives of f^t are given by

$$\begin{aligned} f_1^t &= \Big(1, (-tu_1 - t^2w_1)\cos y, (-tu_1 - t^2w_1)\sin y\Big) + O(t^3)\,, \\ f_2^t &= \Big(0, (-tu_2 - t^2w_2)\cos y - (r - tu - t^2w)\sin y, \\ &\qquad (-tu_2 - t^2w_2)\sin y + (r - tu - t^2w)\cos y\Big) + O(t^3)\,. \end{aligned}$$

For the first fundamental form we obtain

$$g_{11}^t = 1 + t^2 u_1^2 + O(t^3)\,,$$

as well as

$$\begin{aligned} g_{12}^t &= (-tu_1 - t^2 w_1)(-tu_2 - t^2 w_2)\cos^2 y \\ &\quad - (-tu_1 - t^2 w_1)(r - tu - t^2 w)\sin y \cos y \\ &\quad + (-tu_1 - t^2 w_1)(-tu_2 - t^2 w_2)\sin^2 y \\ &\quad + (-tu_1 - t^2 w_1)(r - tu - t^2 w)\sin y \cos y \\ &= t^2 u_1 u_2 + O(t^3)\,, \end{aligned}$$

and

$$\begin{aligned} g_{22}^t &= (-tu_2 - t^2 w_2)^2 \cos^2 y - 2(-tu_2 - t^2 w_2)(r - tu - t^2 w)\sin y \cos y \\ &\quad + (r - tu - t^2 w)^2 \sin^2 y \\ &\quad + (-tu_2 - t^2 w_2)^2 \sin^2 y + 2(-tu_2 - t^2 w_2)(r - tu - t^2 w)\sin y \cos y \\ &\quad + (r - tu - t^2 w)^2 \cos^2 y + O(t^3) \\ &= r^2 - 2rut + (u^2 + u_2^2 - 2rw)t^2 + O(t^3) \end{aligned}$$

This gives for the Gram determinant

$$\det g^t = r^2 - 2rut + (u^2 + u_2^2 - 2rw + r^2 u_1^2)t^2 + O(t^3)\,. \tag{1.21}$$

Now Taylor's formula gives for a function h with $h(0) \neq 0$:

$$\begin{aligned} \big(h(t)\big)^{-5/2} &= h(0)^{-5/2} - \frac{5}{2} h(0)^{-7/2} h'(0) t \\ &\quad + \frac{1}{2}\Big(\frac{35}{4} h(0)^{-9/2}\big(h'(0)\big)^2 - \frac{5}{2}\big(h(0)\big)^{-7/2} h''(0)\Big)t^2 + O(t^3)\,. \end{aligned}$$

Inserting (1.21) gives

$$
\begin{aligned}
(\det g^t)^{-5/2} &= r^{-5} - \frac{5}{2}r^{-7}(-2ru)t \\
&\quad + \frac{1}{2}\Big(35r^{-9}r^2u^2 - 5r^{-7}(u^2 + u_2^2 - 2rw + r^2u_1^2)\Big)t^2 + O(t^3) \\
&= r^{-5} + 5r^{-6}ut + \frac{1}{2}r^{-7}(30u^2 - 5u_2^2 + 10rw - 5r^2u_1^2)t^2 + O(t^3).
\end{aligned} \tag{1.22}
$$

The second partial derivatives of f^t are given by

$$
\begin{aligned}
f^t_{11} &= \Big(0, (-tu_{11} - t^2w_{11})\cos y, (-tu_{11} - t^2w_{11})\sin y\Big) + O(t^3)\,, \\
f^t_{12} &= \Big(0, (-tu_{12} - t^2w_{12})\cos y - (-tu_1 - t^2w_1)\sin y, \\
&\qquad (-tu_{12} - t^2w_{12})\sin y + (-tu_1 - t^2w_1)\cos y\Big) + O(t^3)\,,
\end{aligned}
$$

as well as

$$
\begin{aligned}
f^t_{22} &= \Big(0, (-tu_{22} - t^2w_{22})\cos y - (-tu_2 - t^2w_2)\sin y \\
&\qquad - (-tu_2 - t^2w_2)\sin y - (r - tu - t^2w)\cos y, \\
&\quad (-tu_{22} - t^2w_{22})\sin y + (-tu_2 - t^2w_2)\cos y \\
&\qquad + (-tu_2 - t^2w_2)\cos y - (r - tu - t^2w)\sin y\Big) + O(t^3) \\
&= \Big(0, (-tu_{22} - t^2w_{22})\cos y - 2(-tu_2 - t^2w_2)\sin y - (r - tu - t^2w)\cos y, \\
&\quad (-tu_{22} - t^2w_{22})\sin y + 2(-tu_2 - t^2w_2)\cos y - (r - tu - t^2w)\sin y\Big). \\
&\quad + O(t^3)
\end{aligned}
$$

Denote by $\nu^t = f^t_1 \times f^t_2 / |f^t_1 \times f^t_2|$ a normal to f^t. One easily verifies

$$
\begin{aligned}
\sqrt{\det g^t}\,\nu^t &= \Big((-tu_1 - t^2w_1)(r - tu - t^2w), \\
&\qquad - (-tu_2 - t^2w_2)\sin y - (r - tu - t^2w)\cos y, \\
&\qquad (-tu_2 - t^2w_2)\cos y - (r - tu - t^2w)\sin y\Big) + O(t^3)\,,
\end{aligned}
$$

so the second fundamental form b can be computed as

$$\begin{aligned}
b^t_{11}\sqrt{\det g^t} &= \left\langle f^t_{11}, \nu^t\sqrt{\det g^t}\right\rangle\\
&= -(-tu_{11} - t^2w_{11})(-tu_2 - t^2w_2)\sin y\cos y\\
&\quad - (-tu_{11} - t^2w_{11})(r - tu - t^2w)\cos^2 y\\
&\quad + (-tu_{11} - t^2w_{11})(-tu_2 - t^2w_2)\sin y\cos y\\
&\quad - (-tu_{11} - t^2w_{11})(r - tu - t^2w)\sin^2 y + O(t^3)\\
&= ru_{11}t + (rw_{11} - uu_{11})t^2 + O(t^3)\,,\\
b^t_{12}\sqrt{\det g^t} &= \left\langle f^t_{12}, \nu^t\sqrt{\det g^t}\right\rangle = O(t)
\end{aligned}$$

and

$$\begin{aligned}
b^t_{22}\sqrt{\det g^t} &= \left\langle f^t_{22}, \nu^t\sqrt{\det g^t}\right\rangle\\
&= -(-tu_{22} - t^2w_{22})(-tu_2 - t^2w_2)\sin y\cos y\\
&\quad - (-tu_{22} - t^2w_{22})(r - tu - t^2w)\cos^2 y\\
&\quad + 2(-tu_2 - t^2w_2)^2\sin^2 y\\
&\quad + 2(-tu_2 - t^2w_2)(r - tu - t^2w)\sin y\cos y\\
&\quad + (-tu_2 - t^2w_2)(r - tu - t^2w)\sin y\cos y\\
&\quad + (r - tu - t^2w)^2\cos^2 y\\
&\quad + (-tu_{22} - t^2w_{22})(-tu_2 - t^2w_2)\sin y\cos y\\
&\quad - (-tu_{22} - t^2w_{22})(r - tu - t^2w)\sin^2 y\\
&\quad + 2(-tu_2 - t^2w_2)^2\cos^2 y\\
&\quad - 2(-tu_2 - t^2w_2)(r - tu - t^2w)\sin y\cos y\\
&\quad - (-tu_2 - t^2w_2)(r - tu - t^2w)\sin y\cos y\\
&\quad + (r - tu - t^2w)^2\sin^2 y + O(t^3)\\
&= 2(-tu_2 - t^2w_2)^2 + (r - tu - t^2w)^2\\
&\quad - (-tu_{22} - t^2w_{22})(r - tu - t^2w) + O(t^3)\\
&= r^2 + (ru_{22} - 2ru)t + (u^2 + 2u_2^2 - 2rw - uu_{22} + rw_{22})t^2 + O(t^3)
\end{aligned}$$

For first and second order terms of X^t we obtain

$$\begin{aligned}
g_{22}^t b_{11}^t \sqrt{\det g^t} &= r^3 u_{11} t - 2r^2 u u_{11} t^2 + (r w_{11} - r^2 u u_{11}) t^2 + O(t^3) \\
g_{11}^t b_{22}^t \sqrt{\det g^t} &= r^2 + (r u_{22} - 2ru) t \\
&\quad + (u^2 - u u_{22} + 2u_2^2 - 2rw + r^2 u_1^2 + r w_{22}) t^2 + O(t^3)\,, \\
g_{12}^t b_{12}^t \sqrt{\det g^t} &= O(t^3)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
X^t = {} & r^2 + (r u_{22} - 2ur + r^3 u_{11}) t \\
& + (u^2 - u u_{22} + 2u_2^2 - 2rw + r^2 u_1^2 + r^3 w_{11} + r w_{22} - 3r^2 u u_{11}) t^2 + O(t^3)
\end{aligned}$$

and so

$$\begin{aligned}
(X^t)^2 &= r^4 + 2r^2 (r u_{22} - 2ur + r^3 u_{11}) t + \Big((r u_{22} - 2ur + r^3 u_{11})^2 \\
&\quad + 2r^2 (u^2 - u u_{22} + 2u_2^2 - 2rw + r^2 u_1^2 + r^3 w_{11} + r w_{22} - 3r^2 u u_{11}) \Big) t^2 \\
&\quad + O(t^3) \\
&= r^4 + 2r^2 (r u_{22} - 2ur + r^3 u_{11}) t + r^2 \Big(u_{22}^2 + 4u^2 + r^4 u_{11}^2 - 4u u_{22} \\
&\quad + 2r^2 u_{11} u_{22} - 4r^2 u u_{11} + 2u^2 - 4rw - 2u u_{22} + 2r w_{22} + 4u_2^2 \\
&\quad + 2r^2 u_1^2 + 2r^3 w_{11} - 6r^2 u u_{11} \Big) t^2 + O(t^3)\,.
\end{aligned} \tag{1.23}$$

Combining (1.22) and (1.23) gives

$$\begin{aligned}
(\det g^t)^{-5/2} (X^t)^2 &= r^{-1} + r^{-2} (2u_{22} - 4u + 2r^2 u_{11} + 5u) t \\
&\quad + r^{-3} \Big(u_{22}^2 + 4u^2 + r^4 u_{11}^2 - 4u u_{22} + 2r^2 u_{11} u_{22} \\
&\quad - 4r^2 u u_{11} + 2u^2 - 4rw - 2u u_{22} + 2r w_{22} + 4u_2^2 + 2r^2 u_1^2 \\
&\quad + 2r^3 w_{11} - 6r^2 u u_{11} + 10 u u_{22} - 20u^2 + 10 r^2 u u_{11} \\
&\quad + 15u^2 - \frac{5}{2} u_2^2 - \frac{5}{2} r^2 u_1^2 + 5rw \Big) t^2 + O(t^3)\,.
\end{aligned} \tag{1.24}$$

Since u, w are periodic functions, we have

$$\int_{-\pi}^{\pi}\int_0^{\ell} u_i \,\mathrm{d}x\,\mathrm{d}y = \int_{-\pi}^{\pi}\int_0^{\ell} u_{ij} \,\mathrm{d}x\,\mathrm{d}y = 0\,,$$

as well as

$$\int_{-\pi}^{\pi}\int_0^{\ell} w_i \,\mathrm{d}x\,\mathrm{d}y = \int_{-\pi}^{\pi}\int_0^{\ell} w_{ij} \,\mathrm{d}x\,\mathrm{d}y = 0$$

for $1 \leq i, j \leq 2$. Integration by parts yields

$$\int_{-\pi}^{\pi}\int_0^{\ell} u_{ij} u \,\mathrm{d}x\,\mathrm{d}y = -\int_{-\pi}^{\pi}\int_0^{\ell} u_i u_j \,\mathrm{d}x\,\mathrm{d}y\,.$$

Inserting (1.24) into (1.20) then gives the desired expansion. □

Combining the Willmore energy expansion of the Lemma with the volume constraint (1.18) we can now show the stability of cylinders as periodic Willmore surfaces under a volume constraint:

Theorem 1.3. *Let f as in* (1.16) *parametrize a cylinder and $f^t := f + t u\nu + t^2 w\nu + O(t^3)$ be a volume preserving variation. The second variation of the Willmore energy satisfies*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} W(f^t)\bigg|_{t=0} \geq 0\,.$$

Equality holds if and only if

$$u(x,y) = a\cos(y) + b\sin(y) \tag{1.25}$$

with $a, b \in \mathbb{R}$ and w satisfies (1.18). *In particular, cylinders are stable Willmore surfaces under a volume constraint.*

Note that families of translations (perpendicular to the axis) induce the variation fields (1.25). Since the Theorem does not assert strict stability, it is not immediate that cylinders are local minima of the Willmore energy under a volume constraint, let alone that they are global minima.

Remark 1.4. The Theorem holds under weaker assumptions: The proof does not use that f^t is critical for the volum $V(f^t)$, only the second variation of $V(f^t)$ needs to vanish at $t = 0$. However, in view of the expansions (1.19) and (1.17), if f^t is critical for the Willmore energy, it is critical for the volume anyway.

Proof. Volume preservation of f^t in form of the second equation of (1.18) gives

$$\int_{-\pi}^{\pi}\int_0^{\ell} 2rw \,\mathrm{d}x\,\mathrm{d}y = \int_{-\pi}^{\pi}\int_0^{\ell} u^2 \,\mathrm{d}x\,\mathrm{d}y$$

and so the expansion of the Willmore energy (1.19) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}W(f^t)\bigg|_{t=0} = \frac{1}{4r^3}\int_{-\pi}^{\pi}\int_0^{\ell} 3u^2 - r^2u_1^2 - 5u_2^2 + 4r^2u_{12}^2 + 2r^4u_{11}^2 + 2u_{22}^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{1.26}$$

To show that the right-hand side of (1.26) is non-negative, we use a Fourier series for the doubly periodic function u

$$u(x,y) = \sum_{j,k\in\mathbb{Z}} c_{jk} e^{2\pi i jx/\ell} e^{iky}\,,$$

with coefficients $c_{jk} \in \mathbb{C}$. Applying Parseval's identity to (1.26), we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}W(f^t)\bigg|_{t=0} = \frac{1}{4r^3}\sum_{j,k\in\mathbb{Z}} \Big(|c_{jk}|^2\Big(3 - q^2j^2 - 5k^2 + 4q^2j^2k^2 + 2q^4j^4 + 2k^4\Big)\Big)\,,$$

where we set $q := \frac{2\pi r}{\ell} > 0$. We claim

$$\begin{aligned} s_{jk} &:= 3 - q^2j^2 - 5k^2 + 4q^2j^2k^2 + 2q^4j^4 + 2k^4 \\ &= 3 + q^2j^2(4k^2 + 2q^2j^2 - 1) + k^2(2k^2 - 5) \end{aligned} \tag{1.27}$$

is non-negative for all $j, k \in \mathbb{Z}$ and vanishes if and only if $j = 0$ and $k = 1$, thereby completing the proof. To verify the claim, consider first the case

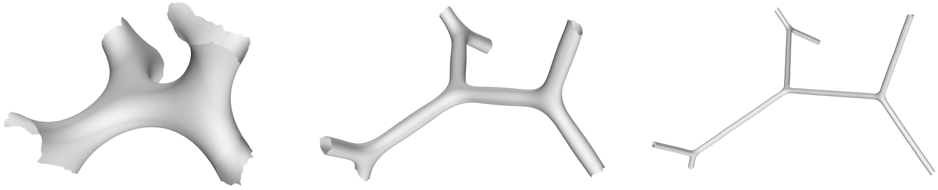


Figure 1.1: Minimizing the Willmore energy of the gyroid under a volume constraint. Decreasing the volume to 10% (left), 1% (center) and 0.1% (right) shows the degeneration of the surfaces to a network, a graph with straight edges.

where $k = 0$. Then (1.27) becomes

$$s_{j0} = 3 + q^2 j^2 (2q^2 j^2 - 1)\,.$$

Clearly, s_{j0} is strictly positive for all $j \in \mathbb{Z}$. Now assume $k^2 \geq 1$. Since

$$q^2 j^2 (4k^2 + 2q^2 j^2 - 1) \geq 0 \tag{1.28}$$

we have

$$s_{jk} \geq 3 + k^2(2k^2 - 5) \geq 0 \tag{1.29}$$

for all $k^2 \geq 1$. Equality in (1.28) holds if and only if $j = 0$ and equality in (1.29) holds if and only if $k = 1$. This proves the claim. □

1.4 Heuristic ideas for periodic Willmore surfaces and related networks

As in the introduction, consider deformations of a periodic surface Σ with lattice Λ obtained by minimizing the Willmore energy under a varying volume constraint. In Surface Evolver computations, see Figure 1.1, with a triply periodic minimal surface to start with, the Willmore minimizers were observed to degenerate into networks with straight edges, the *skeletal*

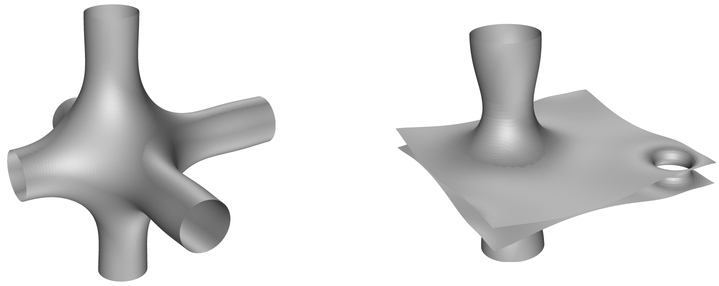


Figure 1.2: Two triply periodic surfaces of the same topological class and the same enclosed volume. Both surfaces are derived from the Schwarz-P surface by minimizing the Willmore energy while continuously reducing the enclosed volume. The left surface degenerates into a cylindrical network. The right surface has less than 60% of the Willmore energy of the left surface. It is comprised of two almost flat planes connected by two channels.

graph of the surface, when the imposed volume constraint approaches 0. For small volumes, the surface pieces corresponding to the edges of the network appear almost cylindrical. For completeness let us note, however, that other geometries are also known to arise, see Figure 1.2; then zero and negative volumes can be attained, in which case embeddedness fails.

To introduce a heuristic reasoning for the cylindrical geometries, let us think of a surface Σ enclosing a small positive volume V_Σ as decomposed into approximately cylindrical pieces joined by junction pieces with boundaries which are almost round circles; e.g., in Figure 1.1 the junction pieces are tripods. As the volume V_Σ tends to 0, the junction pieces approximately scale down, such that the relevant lengths of the cylindrical tubes approach the edge lengths ℓ_i of the network N. Note that the Willmore energy of Σ is the sum of the energy over all tubes and junction pieces.

Since the Willmore energy is scaling invariant, the junction pieces give a constant contribution to the energy, independent of V_Σ. Hence, in a minimizing sequence of surfaces the contribution of the junction pieces to the Willmore energy is uniformly bounded by some constant C_Σ.

On the other hand, the energy of a single exact cylindrical tube with volume $V_i = \pi r_i^2 \ell_i$ is given by $W = \frac{\pi \ell}{2r}$, cf. (1.15). Hence

$$W(\Sigma) \approx \sum_{i=1}^{n} \frac{\pi \ell_i}{2r_i} + C_\Sigma \,,$$

where n denotes the number of tubes and $\ell_i, r_i > 0$ denote their lengths and radii, respectively. Consider two cylindrical pieces of given length ℓ_i with radii r_i chosen such that the total enclosed volume is fixed. A little computation shows that the Willmore energy is minimal precisely when the two radii coincide. Thus, we may assume that for Willmore minimizers under a volume constraint, the radii of all cylindrical pieces must be equal, $r_i = r_\Sigma$ for $i = 1, \ldots, n$. Thus a network with total length

$$L := \sum_{i=1}^{n} \ell_i$$

encloses the volume

$$V_\Sigma \approx \pi r_\Sigma^2 \, L \,,$$

for $V_\Sigma \approx 0$, where it is fair to ignore the junctions. Hence, close to the degenerate limit we obtain

$$W(\Sigma) \approx \sqrt{\frac{\pi^3}{4V_\Sigma} \, L^3} + C_\Sigma$$

for the Willmore energy of Σ. That is, for small V_Σ, the energy of Σ is unbounded and dominated by the network length. We conclude the following for a family of Willmore minimizers under volume constraints: If the surfaces limit in a network as the constraint tends to 0, we expect that *the limiting network minimizes length.* This is a well-posed problem for a fixed lattice.

We are aware that to model the minimization of Willmore energy in any natural context it would be superficial to prescribe a lattice. In fact, this implies that other constraints must be present. Nevertheless, the previous considerations extend to the case of surfaces with a class of lattices. Let Σ,

Σ' be periodic surfaces with lattices Λ, Λ', respectively. Assume Σ, Σ' enclose the same volume $V_{\Sigma'} = V_\Sigma \approx 0$ and that the limiting networks agree as combinatorical graphs. By the above reasoning, the Willmore energy $W(\Sigma')$ is smaller than $W(\Sigma)$ if and only if the length L' of its network is smaller than L. Thus Willmore minimizers for small enclosed volume with the geometry of a tubular neighbourhood will only exist for length minimizing periodic networks. As we shall see, this problem is still well-posed for variable lattices provided we consider only lattices with a fixed volume of the fundamental domain, i.e., $V(\mathbb{R}^3/\Lambda) = V(\mathbb{R}^3/\Lambda')$.

Remark 1.5. The example given in Figure 1.2, pointed out to me by Rob Kusner, suggests that cylindrical networks are not necessarily global minimizers for the Willmore energy in their respective topological class.

2 The Steiner tree problem with boundary vertices

In this chapter we consider the Steiner problem with boundary vertices in Euclidean space: For a given set of points, find a graph of minimal length connecting these points. Minimizers are determined among mappings of a prescribed topological graph. Existence of a minimizer is easily verified; we are interested in its uniqueness. We present a geometric condition equivalent to the uniqueness of a minimizer.

2.1 Finite multigraphs

We only need the notion of simple graphs in this chapter. For the purposes of Chapters 3 and 4, however, it is convenient to give a definition of multigraphs (cf. Figure 2.1).

Definition 2.1. A finite (undirected) *multigraph* G is a pair (V, E) of a finite set V and a mapping $E\colon V \cup \binom{V}{2} \to \{0, 1, \ldots\}$. We call V the *set of vertices*.

Here, $V \cup \binom{V}{2}$ denotes the subsets of V with one or two elements. The number $E(\{v, w\}) \geq 0$ denotes the number of edges connecting the vertices v and w. We say that two vertices $v, w \in V$ are *adjacent* if $E(\{v, w\}) > 0$ and that G has k *loops based at* $v \in V$ if $E(\{v\}) = k$. In this chapter we only consider *simple* graphs, i.e., graphs without loops or multiple edges:

Definition 2.2. A multigraph is called a *(simple) graph* if $E(\{v\}) = 0$ and

Figure 2.1: A simple graph (V, E) (left) and a multigraph (V, F) (right), each on a set of three vertices $V = \{v_1, v_2, v_3\}$. The simple graph is described by the edge set $E = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}$, while the multigraph is described by the function F, whose nonzero values are $F(\{v_1\}) = F(\{v_1, v_2\}) = 1$ and $F(\{v_2, v_3\}) = 2$.

$E(\{v, w\}) \in \{0, 1\}$ for all $v, w \in V$. For simple graphs we write $E := E^{-1}(1)$ and call E the *set of edges.*

We solve the Steiner problem for simple graphs.

Definition 2.3. Suppose $G = (V, E)$ is a finite simple connected graph. Fix a subset $V_F \subset V$ of vertices and an injective mapping $\beta\colon V_F \to \mathbb{R}^n$. A *$G$-network with boundary β* is a mapping $N\colon G \to \mathbb{R}^n$ such that $N(v) = \beta(v)$ for all $v \in V_F$ and edges are mapped onto straight segments, possibly of length 0.

We call V_F the *set of fixed vertices* and $V_M := V \setminus V_F$ the *set of mobile vertices.* For simplicity, we assume that all mobile vertices have at least two neighbours, i.e., adjacent vertices. We denote by $\mathcal{N}_G(\beta)$ the class of all G-networks with boundary β. Finally, the *length* $L(N)$ of a network is given by the sum of its edge lengths.

Our goal is to find a unique network which minimizes length within a prescribed class $\mathcal{N}_G(\beta)$.

2.2 Existence of a minimizer

It is easy to see that for any prescribed class $\mathcal{N}_G(\beta)$ there is at least one network which minimizes length.

Proposition 2.4. *There is a network $N_0 \in \mathcal{N}_G(\beta)$ such that*

$$L(N_0) = \inf \left\{ L(N) \mid N \in \mathcal{N}_G(\beta) \right\}.$$

We call N_0 a minimizer *for $\mathcal{N}_G(\beta)$.*

Proof. A network in $\mathcal{N}_G(\beta)$ is uniquely determined by the placement of the mobile vertices V_M. We can therefore introduce a topology τ on $\mathcal{N}_G(\beta)$ by identifying a network $N \in \mathcal{N}_G(\beta)$ with the placement $N(V_M) \subset (\mathbb{R}^n)^m$, where $m := |V_M|$ is the number of mobile vertices, and using the standard topology on $(\mathbb{R}^n)^m \cong \mathbb{R}^{nm}$. The length

$$L \colon \mathcal{N}_G(\beta) \cong \mathbb{R}^{nm} \to [0, \infty)$$

then is continuous with respect to τ.

Set $I := \inf L$ and let (N_i) be a sequence of networks with $L(N_i) \to I$. We may assume that $L(N_i) < I + 1$ for all i and that $\beta(V_F)$ contains the origin. Then the images of the m mobile vertices $N_i(v_1), \ldots, N_i(v_m)$ are contained in the closed ball of radius $I + 1$ centered at the origin. Therefore there exists a subsequence $N_{i_k}(v_j)$ which converges to some set of points $N_0(v_1), \ldots, N_0(v_m)$ which defines a network N_0. The length L is continuous and so $L(N_0)$ is the limit of the subsequence $L(N_{i_k})$. □

It is possible for a network N to map two adjacent vertices $v, w \in V$ onto the same point in Euclidean space. In this case the corresponding edge $\{v, w\} \in E$ does not contribute to the length of the network.

Definition 2.5. Let $N \in \mathcal{N}_G(\beta)$ be a network. An edge $\{v, w\} \in E$ *vanishes (in N)* if $N(v) = N(w)$. We call a network *degenerate* if at least one edge vanishes. The class of non-degenerate networks is denoted by $N^*_G(\beta)$.

2.3 Variation of networks

To prove uniqueness of minimizers we introduce variations of networks. As in the proof of Proposition 2.4 networks of a prescribed class are determined by the placement of their mobile vertices; any class $\mathcal{N}_G(\beta)$ is isomorphic to $\mathbb{R}^{nm}$. Hence, a differential structure is defined.

Definition 2.6. Let $N \in \mathcal{N}_G(\beta)$ be a network. A *variation of N with variation vectors $b\colon V \to \mathbb{R}^n$* is a family of networks $N_t \in \mathcal{N}_G(\beta)$ such that $N_t(v) = N(v) + tb(v)$ for all mobile vertices $v \in V_M$ and $b(w) = 0$ for all fixed vertices $w \in V_F$.

Essential to the uniqueness theorem is the convexity of the length.

Lemma 2.7. *Let $\ell(t) := |p + tb|$ for some $p, b \in \mathbb{R}^n$ and $p \neq 0$. Then ℓ is convex on $\mathbb{R}$ and twice differentiable at $t = 0$ with*

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell(t)\bigg|_{t=0} = \frac{1}{|p|}\langle p, b\rangle\,, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ell(t)\bigg|_{t=0} = \frac{1}{|p|^3}\Big(|p|^2|b|^2 - \langle p, b\rangle^2\Big)\,. \tag{2.1}$$

In particular, $\ell''(0)$ is non-negative and vanishes if and only if b and p are parallel.

Proof. The triangle inequality applied to $\ell\Big(\lambda t + (1-\lambda)s\Big)$ yields convexity and we have

$$\ell(t) = \sqrt{\langle p + tb, p + tb\rangle} = \sqrt{|p|^2 + 2t\langle p, b\rangle + t^2|b|^2}\,.$$

Therefore, ℓ is smooth at $t = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell(t) = \frac{1}{|p + tb|}\Big(\langle p, b\rangle + t|b|^2\Big)\,.$$

Moreover,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ell(t)\bigg|_{t=0} = \frac{1}{|p|^2}\Big(|p|\,|b|^2 - \frac{1}{|p|}\langle p, b\rangle\langle p, b\rangle\Big) = \frac{1}{|p|^3}\Big(|p|^2|b|^2 - \langle p, b\rangle^2\Big)\,.$$

The Cauchy-Schwarz inequality gives that $\ell''(0)$ is non-negative and vanishes if and only if b and p are parallel. $\square$

Note that for any variation N_t the length of N_t is given by

$$\begin{aligned} L(N_t) &= \sum_{\{v,w\}\in E} \Big|\Big(N(v)+tb(v)\Big)-\Big(N(w)+tb(w)\Big)\Big| \\ &= \sum_{\{v,w\}\in E} \Big|\Big(N(v)-N(w)\Big)+t\Big(b(v)-b(w)\Big)\Big|\,. \end{aligned}$$

Hence, the function $t\mapsto L(N_t)$ is the sum of $|E|$ functions of type $t\mapsto |p+tb|$. Setting $p := N(v)-N(w)$ and $b := b(v)-b(w)$ in (2.1) immediately gives:

Lemma 2.8. *Let N_t be a variation with variation vectors b of some non-degenerate network $N \in \mathcal{N}^*_G(\beta)$. Then, the length along the variation $t\mapsto L(t) := L(N_t)$ is convex, twice differentiable at $t=0$ and the variation formulas*

$$\begin{aligned} L'(0) &= \sum_{\{v,w\}\in E} \frac{1}{|N(v)-N(w)|}\langle N(v)-N(w), b(v)-b(w)\rangle\,, \\ L''(0) &= \sum_{\{v,w\}\in E} \frac{|N(v)-N(w)|^2|b(v)-b(w)|^2-\langle N(v)-N(w), b(v)-b(w)\rangle^2}{|N(v)-N(w)|^3} \end{aligned} \tag{2.2}$$

hold. In particular, $L''(0)$ is non-negative and $L''(0)=0$ holds if and only if

$$b(v)-b(w) \parallel N(v)-N(w) \qquad \textit{for all } \{v,w\}\in E\,. \tag{2.3}$$

We call N *critical* if $L'(0)=0$ for all variations of N. By convexity of the length, a non-degenerate network is a minimizer if and only if it is critical.

There is a physical interpretation of criticality: Consider the *force vector* exerted at each vertex,

$$F(v) = \sum_{\{v,w\}\in E} \frac{N(v)-N(w)}{|N(v)-N(w)|}\,,$$

which is the sum of unit vectors pointing in the edge directions. Then Lemma 2.8 asserts that $L'(0) = 0$ for all variations N_t if and only if

$$F(v) = 0\,, \qquad \text{for all } v \in V_M\,. \tag{2.4}$$

Networks satisfying (2.4) are called *balanced*. We have obtained:

Proposition 2.9. *For a non-degenerate network $N \in \mathcal{N}^*_G(\beta)$ the following are equivalent:*

(i) N is balanced.

(ii) N is critical.

(iii) N is a minimizer for $\mathcal{N}_G(\beta)$.

2.4 Uniqueness of minimizers for prescribed topology

In this section we characterize uniqueness of minimizers for prescribed graphs. Ivanov and Tuzhilin [IT94] prove the uniquenss of minimizers under the assumption that the minimizer is embedded:

Proposition ([IT94, Theorem 3.1])**.** *Let $N \in \mathcal{N}^*_G(\beta)$ be an embedded minimizer, where G is a tree and V_M contains no vertices of degree 2. Then N is the unique minimizer in $\mathcal{N}_G(\beta)$.*

We provide a generalization of the uniqueness theorem and characterize uniqueness of (possibly degenerate) minimizers in the class $\mathcal{N}_G(\beta)$. If G has cycles, then the hexagonal network as in Figure 2.2 shows that uniqueness of minimizers cannot be expected. So we will assume that G is a tree. We find it instructive to consider the non-degenerate case first.

Since non-degenerate minimizers are balanced, at a vertex of degree 2 the two incident edges are directed oppositely. Then, $N(v)$ can be moved along the straight segment connecting its neighbours without changing the length

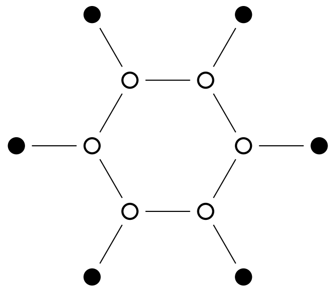


Figure 2.2: A network with a non-unique minimizer. The black vertices are fixed and placed at the vertices of a regular hexagon with radius 1. For any $0 < r < 1$ the white vertices can be placed at the vertices of a regular hexagon with radius r without changing length. The resulting network is balanced, thus a minimizer. The network is degenerate for $r \in \{0, 1\}$.

of N. Uniqueness of minimizers in the class $\mathcal{N}_G^*(\beta)$ cannot be expected if G contains mobile vertices of degree 2. The same argument holds if a balanced vertex v of any degree is placed such that its neighbours are collinear.

Lemma 2.10. *Suppose G is a tree. Let $N \in \mathcal{N}_G^*(\beta)$ be a non-degenerate balanced network. There is a non-zero varation N_t of N such that* (2.3) *holds if and only if there is a mobile vertex $v \in V_M$ such that $N(v)$ is collinear with all its neighbours.*

Proof. „$\Leftarrow$“ First assume there is a mobile vertex $v \in V_M$ such that $N(v)$ is collinear with all its neighbours. We define the variation N_t as follows: set $b(v)$ to a unit vector pointing in the common edge direction at $N(v)$ and set all other values of b to 0. Since N is balanced at any vertex $v \in V_M$, at v the number of edges pointing in the same direction as $b(v)$ is the same as the number of edges opposite to $b(v)$. So clearly (2.3) is satisfied.

„$\Rightarrow$“ Now suppose there is no vertex collinear with its neighbours, in particular all vertices have degree $d \geq 3$. Consider a non-zero variation N_t with variation vectors $b\colon V \to \mathbb{R}^n$, that is,

$$V^* := \left\{ v \in V_M \mid b(v) \neq 0 \right\}$$

is non-empty. Denote by G^* the induced subgraph of G consisting of the vertices V^*. Let $G_0 \subset G^*$ be a connected component of G^*. Since G is a tree, G_0 is also a tree.

Suppose first G_0 consists of a single vertex v_0. Since N is balanced, v_0 has at least three neighbours w_1, w_2, w_3 not in G_0, thus $b(w_i) = 0$ for $i = 1, 2, 3$. Since the edges $N(w_i) - N(v)$ incident to $N(v)$ are not collinear, the three vectors $b(v_0) - b(w_i) = b(v_0)$, $i = 1, 2, 3$, cannot satisfy (2.3).

Now assume G_0 consists of at least two vertices. Choose a leaf $v_0 \in G_0$, that is, a vertex with one neighbour v_1 in G_0. If v_0 has at least three neighbours in $V \setminus V^*$, by the above reasoning (2.3) cannot hold. So assume there are just two vertices $w_1, w_2 \in V \setminus V^*$ incident to v_0 with $b(w_1) = b(w_2) = 0$. If condition (2.3) holds at v_0, the two edges $N(w_i) - N(v_0)$, $i = 1, 2$, must be parallel to $b(v_0) - b(w_i) = b(v_0)$. In particular, $N(v_0), N(w_1), N(w_2)$ must be collinear. On the other hand, by assumption $N(v_0)$ is not collinear with all its neighbours. Hence, at $N(v_0)$ all edges but one are collinear. Then the network N cannot be balanced at v_0. So this case cannot arise. □

Proposition 2.11. *Suppose G is a tree and $N_0 \in \mathcal{N}_G^*(\beta)$ is a minimizer. Then N_0 is the unique minimizer for $\mathcal{N}_G(\beta)$ if and only if no mobile vertex of N_0 is collinear with its neighbours.*

Proof. „⇒“ Indirectly, assume there is a mobile vertex $v \in V_M$ such that $N_0(v)$ is collinear with its neighbours. By Lemma 2.10 there is a variation N_t of N with variation vectors b such that condition (2.3) holds. In particular, N_t is another minimizer in $N_G(\beta)$ for small $t > 0$. So N is not a unique minimizer.

„⇐“ For the converse suppose there is another, possibly degenerate minimizer $N_1 \in \mathcal{N}_G(\beta)$. Consider the variation N_t given by the variation vectors $b(v) := N_1(v) - N_0(v)$. The length $L(N_t)$ is convex and critical at $t = 0$. Since $L(N_1) = L(N_0)$, $L(N_t)$ is constant on $[0, 1]$ by Lemma 2.8. In particular, (2.3) holds for $t \in [0, 1]$. By Lemma 2.10 there exists a vertex $v \in V_M$ collinear with its neighbours. □

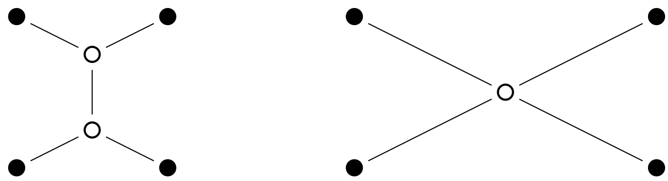


Figure 2.3: A graph G (left) and the degenerate minimizer N (right) mapping G into Euclidean space. The black vertices are fixed and placed at the vertices of a long rectangle. The white vertices are both placed at the center of the rectangle.

In the case of Proposition 2.11 where a minimizer N is not unique, we have shown that there is a contiuous family N_t of minimizers. We will use this later in the proof of Theorem 2.13.

2.5 Degenerate minimizers for prescribed topology

In this section we generalize Proposition 2.11 to networks N where the set of vanishing edges

$$E_0 := \Big\{\{v,w\} \in E : N(v) = N(w)\Big\}$$

is not necessarily empty. We wish to assert the uniqueness of minimizers $N \in \mathcal{N}_G(\beta)$. Figure 2.3 shows a graph where the unique minimizer is degenerate and Proposition 2.11 cannot be applied.

We introduce an equivalence relation $\sim$ on the set of vertices as follows (cf. Figure 2.4): Two vertices $v, w \in V$ are equivalent if they are connected by an edge of vanishing length, i.e.,

$$v \sim w \quad :\Longleftrightarrow \quad v \text{ and } w \text{ are connected in } (V, E_0)\,.$$

Clearly, two equivalent vertices $v \sim w$ have the same image $N(v) = N(w)$.

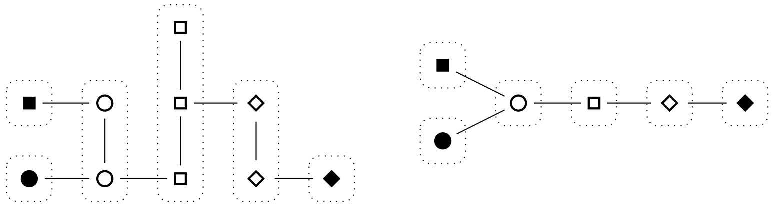


Figure 2.4: A tree G (left) mapped by a degenerate network N and its quotient $\widetilde{G}$ (right). Vertices with the same symbol have the same image under N. Dotted boxes represent the equivalence classes of vertices.

Setting $\widetilde{V} := V/{\sim}$ and

$$\widetilde{E} := \left\{ \{\widetilde{v}, \widetilde{w}\} \in \binom{\widetilde{V}}{2} \mid \{v_0, w_0\} \in E \setminus E_0 \text{ for some } v_0 \in \widetilde{v} \text{ and } w_0 \in \widetilde{w} \right\}$$

yields a graph $\widetilde{G} := (\widetilde{V}, \widetilde{E})$. Here, $\widetilde{v}$ denotes the equivalence class of v. Note that the set of vanishing edges depend on the network N.

We define the *quotient network* $\widetilde{N}$ associated to $\widetilde{G}$ by setting $\widetilde{N}(\widetilde{v}) := N(v)$ for all vertices $v \in V$. By construction, no edge in $\widetilde{E}$ vanishes in $\widetilde{N}$. So $\widetilde{N}$ is a non-degenerate $\widetilde{G}$-network with boundary $\widetilde{\beta}$, where we set $\widetilde{\beta}(\widetilde{w}) := \beta(w)$ for all fixed vertices $w \in V_F$. Since G is a tree and the equivalence classes in V are connected, for any two equivalence classes $\widetilde{v}, \widetilde{w} \in \widetilde{V}$ there is at most one edge $\{v_0, w_0\} \in E \setminus E_0$ with $v_0 \in \widetilde{v}$ and $w_0 \in \widetilde{w}$. Consequently, the lengths of $\widetilde{N} \in \mathcal{N}_{\widetilde{G}}(\widetilde{\beta})$ and $N \in \mathcal{N}_G(\beta)$ are equal and $\widetilde{G}$ is a simple graph. Moreover, since networks with prescribed topology are determined by the placement of their mobile vertices, a G-network can be recovered from its quotient network; that is, if two networks N_0, N_1 have the same set of vanishing edges, then $N_0 = N_1$ if and only if $\widetilde{N_0} = \widetilde{N_1}$. We obtain:

Lemma 2.12. *Suppose G is a tree and $N_0 \in \mathcal{N}_G(\beta)$ is a minimizer. Then $\widetilde{N_0}$ is a minimizer in $\mathcal{N}^*_{\widetilde{G}}(\widetilde{\beta})$.*

Proof. Let $\widetilde{N_1} \in \mathcal{N}_{\widetilde{G}}(\widetilde{\beta})$ be any $\widetilde{G}$-network with boundary $\widetilde{\beta}$. We claim $L(\widetilde{N_1}) \geq L(\widetilde{N_0})$.

We define a network $N_1 \in \mathcal{N}_G(\beta)$ by setting $N_1(v) := \widetilde{N_1}(\widetilde{v})$ for all

$v \in V$. By assumption, the network N_0 minimizes length in $\mathcal{N}_G(\beta)$, i.e., $L(N_1) \geq L(N_0)$. The graph G is a tree so we have, as asserted before,

$$L(\widetilde{N}_1) = L(N_1) \geq L(N_0) = L(\widetilde{N}_0)\,,$$

which proves the claim. □

Using Proposition 2.11, the lemma asserts that a minimizer $N \in \mathcal{N}_G(\beta)$ is unique among all networks with the same set of vanishing edges.

More generally, we want to study if any two minimizers $N_0, N_1 \in \mathcal{N}_G(\beta)$ agree. Consider the variation N_t given by the variation vectors $b(v) := N_1(v) - N_0(v)$. By convexity of $t \mapsto L(N_t)$, all networks N_t with $t \in [0,1]$ are minimizers. Thus Lemma 2.12 gives $N_1 = N_0$ provided there are two networks N_ξ, N_τ with $0 < \xi < \tau < 1$ which have the same set of vanishing edges. This can indeed be shown under the collinearity condition we stated before, and so proves the uniqueness of minimizers in $\mathcal{N}_G(\beta)$ given the condition:

Theorem 2.13. *Suppose G is a tree and $N_0 \in \mathcal{N}_G(\beta)$ is a minimizer. Then N_0 is the unique minimizer in $\mathcal{N}_G(\beta)$ if and only if no mobile vertex of the quotient network $\widetilde{N}_0$ is collinear with its neighbours.*

Proof. „⇒“ Assume there is a mobile vertex in the quotient network $\widetilde{N}_0 \in \mathcal{N}_{\widetilde{G}}(\widetilde{\beta})$ collinear with its neighbours. By Lemma 2.12, $\widetilde{N}_0$ is a minimizer. Proposition 2.11 yields another minimizer $\widetilde{N}_1 \in \mathcal{N}_{\widetilde{G}}(\widetilde{\beta})$ with $\widetilde{N}_1 \neq \widetilde{N}_0$. We define a network $N_1 \in \mathcal{N}_G(\beta)$ by setting $N_1(v) := \widetilde{N}_1(\widetilde{v})$ for all $v \in V$. Since $\widetilde{N}_0 \neq \widetilde{N}_1$, there is a vertex $\widetilde{v} \in \widetilde{V}$ such that $\widetilde{N}_0(\widetilde{v}) \neq \widetilde{N}_1(\widetilde{v})$, thus $N_0(v) \neq N_1(v)$. Moreover, since G is a tree and $\widetilde{N}_0$ and $\widetilde{N}_1$ are minimizers, we have

$$L(N_0) = L(\widetilde{N}_0) = L(\widetilde{N}_1) = L(N_1)\,.$$

This shows $N_1 \in \mathcal{N}_G(\beta)$ is a minimizer, different from N_0.

„⇐“ Assume there is no mobile vertex in the quotient network $\widetilde{N}_0$ collinear with its neighbours but suppose there is another minimizer $N_1 \in \mathcal{N}_G(\beta)$.

Consider the variation N_t of N_0 given by the variation vectors $b(v) := N_1(v) - N_0(v)$. We show $b \equiv 0$, thereby proving $N_1 = N_0$.

Let $e := \{v, w\} \in E$ be an edge. Its length ℓ_e in N_t is given by

$$t \mapsto \ell_e(t) = |N_t(v) - N_t(w)| = \Big|N_0(v) - N_0(w) + t\big(b(v) - b(w)\big)\Big| .$$

Suppose $\ell_e(t) = 0$ for some $t \in [0, 1]$. Since $\ell_e(t)$ is the modulus of an affine function, either ℓ_e vanishes identically on $[0, 1]$ or $\ell_e(t) = 0$ holds on $[0, 1]$ precisely at a single point. Note that the set of edges is finite. Hence, there are $0 < \xi < \tau < 1$ such that for all $e \in E$ the edge $\ell_e(\xi)$ vanishes if and only if $\ell_e(\tau)$ vanishes and, by choosing ξ sufficiently small, no mobile vertex of $\widetilde{N_\xi}$ is collinear with its neighbours. Since the length $t \mapsto L(N_t)$ is convex, N_ξ and N_τ are minimizers in $\mathcal{N}_G(\beta)$. By construction, N_ξ and N_τ have the same set of vanishing edges. Hence, the two networks yield the same quotient graph $\widetilde{G}$. By Lemma 2.12, the quotient networks $\widetilde{N_\xi}$ and $\widetilde{N_\tau}$ are two minimizers in $\mathcal{N}^*_{\widetilde{G}}(\widetilde{\beta})$. Using Proposition 2.11, we obtain $\widetilde{N_\xi} = \widetilde{N_\tau}$. Hence, $N_\xi = N_\tau$ and so $b \equiv 0$. This shows $N_0 = N_1$ and thereby concludes the proof. □

2.6 The triangular Steiner tree problem

The triangular Steiner tree problem considers an elementary geometric question: *Given a triangle $\triangle$, which points S minimize the sum of the distances from S to the vertices of $\triangle$?*

The following theorem answers this question. The theorem is well-known but it is important for the Steiner tree problem with variable topology considered in the next section and for our study of the periodic Steiner tree problem. We denote by $\triangle(A, B, C)$ the convex hull of three non-collinear points A, B, C in Euclidean n-space, and, e.g., by $\angle(A, B, C)$ the interior angle of $\triangle(A, B, C)$ at B.

Theorem 2.14. *Let $\triangle(A, B, C)$ be a triangle. Then there is a unique point $S \in \triangle(A, B, C)$ minimizing $L(P) := |\overline{AP}| + |\overline{BP}| + |\overline{CP}|$.*

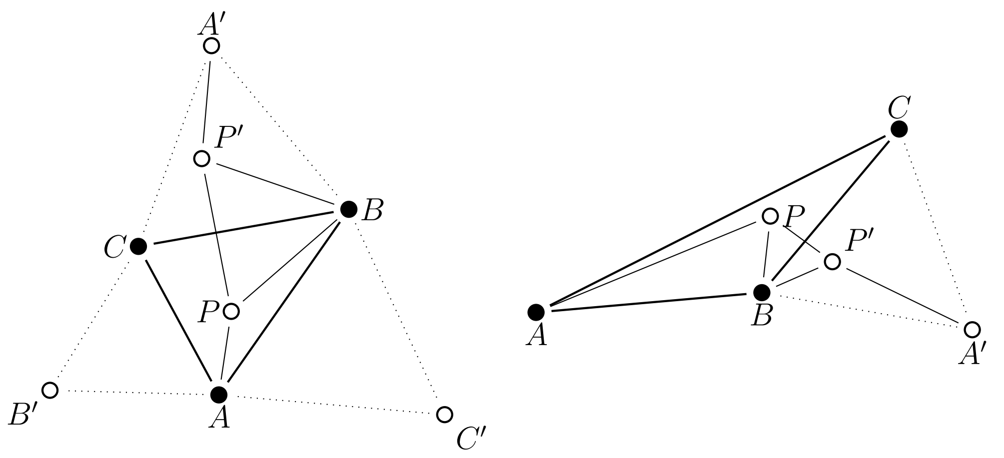


Figure 2.5: Construction of Theorem 2.14. If all interior angles are less than 120 degrees (left), the minimizer is contained in $\triangle(A,B,C)$. If there is an angle of at least 120 degrees (right), the L is minimal at the obtuse angled vertex.

(i) If all interior angles of $\triangle(A,B,C)$ *are less than* 120 *degrees, then*

$$\angle(A,S,B) = \angle(B,S,C) = \angle(C,S,A) = 120^\circ\,.$$

(ii) If there is an interior angle of $\triangle(A,B,C)$ *of at least* 120 *degrees then* S *lies at the obtuse angled vertex.*

We call the point S *Fermat point* of the triangle $\triangle(A,B,C)$. We present a well-known elementary geometric proof which is constructive.

Proof. We may assume that L attains its minimal value at some point in $\triangle(A,B,C)$. Let P be an arbitrary point in the interior of $\triangle(A,B,C)$. Complete the edge $\overline{BP}$ to an equilateral triangle $\triangle(B,P,P')$ such that P' and A lie on different sides of $\overline{BP}$. Also, complement $\overline{BC}$ to an equilateral triangle $\triangle(A',B,C)$ such that A' and A lie on different sides of $\overline{BC}$. Similarly to $\triangle(A',B,C)$, construct equilateral triangles $\triangle(A,B',C)$ and $\triangle(A,B,C')$ (see Figure 2.5). Then, by construction, $|\overline{BC}| = |\overline{BA'}|$ and

$$|\overline{BP}| = |\overline{PP'}| = |\overline{BP'}|\,. \tag{2.5}$$

Moreover, $\angle(P, B, P') = \angle(C, B, A') = 60°$. So the triangles $\triangle(P, B, C)$ and $\triangle(P', B, A')$ are congruent. In particular, we have

$$\angle(B, P, C) = \angle(B, P', A') \quad \text{and} \quad |\overline{CP}| = |\overline{A'P'}|\,. \tag{2.6}$$

Using (2.5) and (2.6) we obtain

$$L(P) = |\overline{AP}| + |\overline{BP}| + |\overline{CP}| = |\overline{AP}| + |\overline{PP'}| + |\overline{P'A'}|\,. \tag{2.7}$$

The triangle inequality yields

$$L(P) \leq |\overline{AA'}|\,,$$

where equality holds if and only if P and P' are contained in $\overline{AA'}$. This is the case if and only if

$$\angle(A, P, B) = 120° \quad \text{and} \quad \angle(B, P', A') = \angle(B, P, C) = 120°\,. \tag{2.8}$$

(i): Assume that all interior angles of $\triangle(A, B, C)$ are less than 120 degrees. Then, the lines $\overline{BB'}$ and $\overline{CC'}$ intersect at a point S in the interior of $\triangle(A, B, C)$. Since $|\overline{AB'}| = |\overline{AC}|$, $|\overline{AB}| = |\overline{AC'}|$ and $\angle(B', A, B) = \angle(C, A, C')$, the triangles $\triangle(B', A, B)$ and $\triangle(C, A, C')$ are isometric. A rotation by 60 degrees about A is an isometry mapping $\triangle(B', A, B)$ onto $\triangle(C, A, C')$. Hence, $\angle(B', S, C) = \angle(C', S, B) = 60°$. Following the construction from above for S we obtain S' and prove that S satisfies (2.8), thus minimizing L.

So let S' be the point which complements $\overline{BS}$ to an equilateral triangle such that S' and A lie on different sides of $\overline{BS}$. From $\angle(B', S, C) = \angle(S', S, B) = 60°$ it follows that $\angle(C, S, S') = \angle(A, S, C') = 60°$. Hence, $\angle(A, S, B) = \angle(B, S, C) = 120°$. By (2.8), S minimizes L and so it is contained in $\overline{AA'}$.

The symmetry of this reasoning in A, B, C implies that any minimizer of L has S contained in $\overline{AA'} \cap \overline{BB'} \cap \overline{CC'}$. This gives the uniqueness of the

Figure 2.6: Two different minimizers for the Steiner tree problem with variable topology. The black vertices are fixed.

minimizer.
(ii): Now assume that $\triangle(A, B, C)$ has an interior angle of of at least $120°$, for instance at B. Then, $\overline{AA'}$ does not intersect the interior of $\triangle(A, B, C)$ and the polygonal chain $\mathcal{P} := \overline{APP'A'}$ lies above the chain $\overline{ABA'}$ with respect to the axis $\overline{AA'}$. Hence, projecting $\mathcal{P}$ onto the triangle $\triangle(A, B, A')$ reduces length if P or P' lie in the interior of $\triangle(A, B, C)$. By (2.7) we obtain

$$|\overline{AB}| + |\overline{BA'}| \leq |\overline{AP}| + |\overline{PP'}| + |\overline{P'A'}| = L(P)\,,$$

where equality holds if and only if $P = P' = B$. $\square$

2.7 Minimizers for variable topology

We conclude this chapter with an existence theorem for minimizers for the Steiner tree problem in the case where the underlying topological graph is not fixed.

A minimizer for the Steiner tree problem with free topology is in general not unique. The example given in Figure 2.6 shows that there are two minimizers for the same boundary condition. The minimizers do not reproduce the full symmetry of the boundary vertices.

Definition 2.15. Let $F \subset \mathbb{R}^n$ be a finite set of points. A network $N \in N_G(\beta)$ is called a *minimal Steiner tree* if $F = \beta(V_F)$ and it minimizes length among all underlying connected abstract graphs G and boundary β.

We call the mobile vertices of N *Steiner points* and we call a vertex v *removable in* N if it has degree 2 and $N(v)$ lies on the edge connecting its two neighbours.

Lemma 2.16. *Let $N\colon G \to \mathbb{R}^n$ be a non-degenerate network with length $L(N)$. Suppose there is a non-removable mobile vertex with degree $d \neq 3$. Then there is a network $N'\colon G' \to \mathbb{R}^n$ with $N(V) \subset N'(V')$ and $L(N') < L(N)$. Moreover, G' has only removable vertices and vertices of degree 3.*

Proof. If the assumption on the degree hold, we will modify N in finitely many steps to obtain N'. Clearly we can decrease length by successive removing all vertices of degree $d = 1$ from N, together with their incident edges.

A degree $d \geq 4$ at a vertex v of the underlying graph can be reduced in following way: There are two vertices w_1, w_2 adjacent to v such that at $N(v)$ the two edges with endpoints $N(w_1), N(w_2)$ make an angle of less than 120 degrees but are not collinear. To redefine N, we insert another mobile vertex v' into G. We replace the two edges $\{w_1, v\}, \{w_2, v\}$ with the three edges $\{w_1, v'\}, \{w_2, v'\}, \{v, v'\}$ and set $N(v') = S$, where S is the fermat point of the triangle $N(w_1), N(w_2), N(v)$ (see Theorem 2.14 and Figure 2.7). This reduces length and decreases the degree of v from d to $d - 1$. Upon reiteration we can reduce the degree to $d \leq 3$ at all vertices.

Similarly, if the resulting network contains a vertex of degree 2 with non-opposite edges we can replace these edges by a single edge to reduce length. □

Theorem 2.17. *For each set $F \subset \mathbb{R}^n$ there exists an embedded minimal Steiner tree with at most $2|F| - 2$ non-removable vertices.*

Proof. Consider a length minimizing sequence (N_k), i.e., $\lim_{k\to\infty} L(N_k) = \inf L(N)$, where the infimum is taken over networks with $N(V_F) = F$. Denote by (G_k) their underlying graphs.

Lemma 2.16 asserts that we can assume that each mobile vertex of G_k has degree $d = 3$ or $d = 2$ with opposite edges. Moreover, we can assume that G_k is a tree. Otherwise we can remove edges, thus decrease length, without changing connectivity. We also remove all removable vertices: we

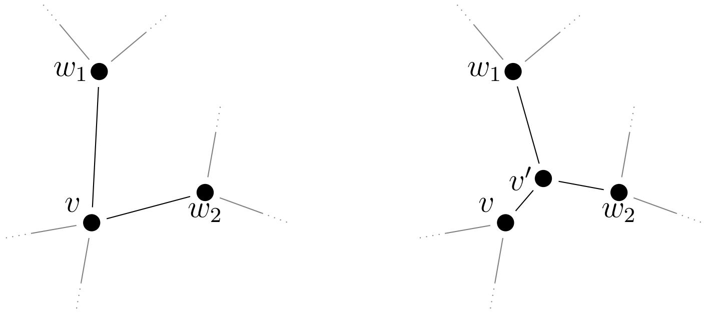


Figure 2.7: An immersed graph with a vertex v of degree $d = 4$ (left) can be shortened by inserting a new vertex v' (right). The vertex v' is placed at the Fermat point of a triangle v, w_1, w_2 which encloses an angle of less than 120 degrees at v'.

replace each pair of opposite edges of (N_k) incident to the same vertex by a single edge, leaving the length unchanged. We claim that G_k now has at most $2|F| - 2$ vertices.

Each mobile vertex $v \in G_k$ has three incident edges, and there are at least $|V_F|$ edges incident to the points in F. Since this count includes edges twice, there are at least $(3\,|V_M| + |V_F|)/2$ edges. Since N is a tree on $|V_M| + |V_F|$ vertices, it has $|V_M| + |V_F| - 1$ edges. We obtain the desired inequality

$$3|V_M| + |V_F| \leq 2|V_M| + 2|V_F| - 2\,.$$

We conclude $|V_M| \leq |V_F| - 2$. This proves the claim.

Consequently, given F, the number of abstract graphs G_k with at most $2|F| - 2$ vertices is finite. By passing to a subsequence we can assume that G_k is constant. Then the existence of a minimizer $N := \lim_k N_k$ follows from Proposition 2.4.

We claim that all edges of N attain positive length. If on the contrary an edge attains length 0, a vertex with degree $d \geq 4$ results. By Lemma 2.16, N cannot be a minimizer of length over all topological types, contradicting the fact that (N_k) is a minimizing sequence.

A similar argument shows N is embedded. An intersection point means that N can be regarded as a length minimizing network with a vertex of degree $d \geq 4$, contradicting again Lemma 2.16. $\square$

The Steiner problem can be generalized to arbitrary complete Riemannian manifolds. The local geometry of minimizers does not change: Criticality and balancing are equivalent regardless of the metric and minimizers have only vertices of degree 3, see [IT94, Theorem 2.1]. However, global assertions on the uniqueness of minimizing networks can fail: A geodesic between two points on a Riemannian manifold is neither unique nor length minimizing.

3 Periodic networks minimizing length

In this chapter we prove the main theorem. We determine the topological type of length minimizing periodic networks and identify the minimal triply periodic network as the srs network:

Theorem 3.1. *Among n-periodic networks with lattice $\Lambda_0 \subset \mathbb{R}^n$, there exists a network which minimizes the length quotient L^n/V.*

(i) Any minimizing network $N\colon G \to \mathbb{R}^n$, after removal of vertices of degree 2 with opposite edges, is a Steiner network such that G_0 has $2n-2$ vertices.

(ii) The length and volume of a triply periodic network N in $\mathbb{R}^3$ satisfy

$$\frac{L^3}{V} \geq \frac{27}{\sqrt{2}} = 19.09\ldots . \tag{3.1}$$

Equality holds exactly for the srs *network, where the lattice Λ_0 is body-centered cubic.*

The terminology is explained in Section 3.1; in particular uniqueness is always up to (unnecessary) vertices of degree 2.

Let us indicate how the results of this chapter combine to prove Theorem 3.1. First, for a fixed lattice we establish the existence of a minimizer of L^n/V in Theorem 3.9. It must be an embedded Steiner network on $2n-2$ vertices. This is part (i) of Theorem 3.1. Since a minimizer cannot have loops in the quotient graph (Lemma 3.11) only the two graphs shown in Figure 3.2 can arise in dimension $n = 3$. Theorems 3.19 and 3.23 then give sharp estimates for the quotient L^3/V of networks with arbitrary lattice for the two cases of quotient graphs. These estimates imply (3.1) as well as the characterization of the equality case. This establishes part (ii) of Theorem 3.1 for arbitrary triply periodic networks.

An obvious approach to prove the estimates of Theorems 3.19 and 3.23 would be to minimize the quotient L^3/V for a given lattice, and thereafter minimize over all lattices. However, Steiner networks for a given lattice are not unique, and lattices are inconvenient to parameterize. So we use a different approach: In Lemmas 3.17 and 3.21 we show that it is possible to parameterize the space of Steiner networks covering each of the underlying graphs by their six edge lengths alone (plus an angle parameter in one case). Then not only the length L but also the volume V become explicit functions of these parameters (Lemma 3.17 and 3.21). Thus to prove Theorems 3.19 and 3.23 we need to solve a finite dimensional optimization problem under constraints: On our parameter space, we minimize the total length L under the constraints that $V = 1$ and the length parameters be positive. Then it turns out that lattice generators are linear functions of our parameters.

3.1 Topological crystals and periodic networks

For our purposes, it is convenient to use the term network in the following sense:

Definition 3.2. An *n-periodic network* $N\colon G \to \mathbb{R}^n$ (with base G_0) is an immersion of a connected simple graph G into $\mathbb{R}^n$, where edges are mapped onto straight segments of positive length, subject to the following:

- $N(G)$ is invariant under a maximal lattice Λ of rank n.
- The quotient $N(G)/\Lambda \subset \mathbb{R}^n/\Lambda$ is the image of the finite connected multigraph G_0, possibly with loops and multiple edges.

We call $V = V(\mathbb{R}^n/\Lambda)$ the *(spanned) volume* of N and $L = L(N(G)/\Lambda)$ the *length* of N.

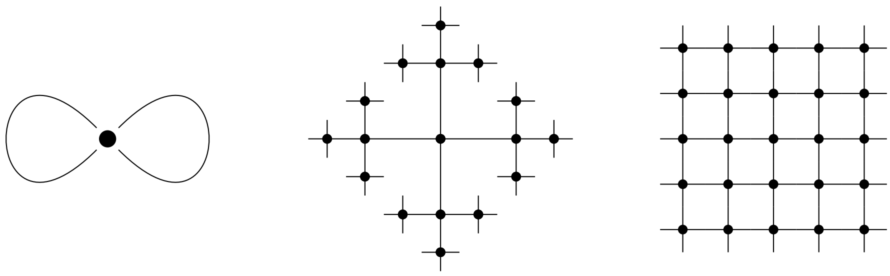


Figure 3.1: A finite multigraph (left), its universal covering graph (center) and its maximal abelian covering graph (right).

Recall that a *lattice* of rank n is a set $\Lambda = \{\sum_{i=1}^{n} a_i g_i : a_i \in \mathbb{Z}\} \subset \mathbb{R}^n$, where the vectors $g_1, \dots, g_n \in \mathbb{R}^n$ are linearly independent. We call Λ *maximal (for N)* if $\Lambda' \subset \Lambda$ for all lattices Λ' for which $N(G) = N(G) + \Lambda'$. The maximal lattice is unique and the ambient space quotient $\mathbb{R}^n/\Lambda$ is an n-dimensional flat torus. It can be represented by a parallelepiped spanned by the vectors g_i. We refer to the quotient or its representing epiped as a *fundamental domain*. The induced mapping $N/\Lambda\colon G_0 \to \mathbb{R}^n/\Lambda$ of the base graph is called the *quotient network*. A network is *immersed* if the star of each vertex is embedded. Here the *star* of a vertex p, denoted $\operatorname{star} p$, is the union of the edges from p to its incident vertices. Clearly, the immersion condition implies simplicity of the network.

Conversely, we can start with an abstract finite base graph G_0. Our networks can then be described as immersions of maximal abelian coverings of G_0 (cf. Figure 3.1). See Sunada [Sun12] for a detailed account of the covering theory of graphs.

Note that an immersed network can have a non-immersed quotient: For instance, in $\mathbb{R}^2$ we consider the network N with lattice $\mathbb{Z}^2$ which is the orbit of the three edges from the origin to $(0,1)$, $(1,0)$ and to $(1,3)$. Then the star of a vertex has only a self-intersection for $N/\mathbb{Z}^2$, not for N.

We wish to minimize the length $L = L(N/\Lambda)$ of the quotient network N/Λ, subject to the constraint that the n-dimensional volume $V = V(\mathbb{R}^n/\Lambda)$ of a fundamental domain is fixed to 1. Equivalently, we may minimize the scaling-invariant *length quotient* L^n/V.

Definition 3.3. An *(n-periodic) minimizer* is a network N that minimizes the length quotient among all n-periodic networks.

Variations of networks as in Definition 2.6 are well-defined for periodic networks. A variation N_t of a periodic network N, however, is in general not periodic. In particular, since N_t is an infinite graph, length and volume of a variation are not defined. Therefore, we require variations to be invariant under the maximal lattice of N.

Definition 3.4. A *(periodic) variation of a periodic network N with lattice* Λ_0 is a family of periodic networks N_t with lattice Λ_0 such that $N_t(v) = N(v) + tb(v)$ with some variation vectors $b\colon V \to \mathbb{R}^n$.

Periodic variations can be understood as variations of the (finite) quotient network. The variation formulas (2.2) then hold for periodic variations. Since the length quotient depends on the lattice Λ, critical networks are not minimizers in general, i.e., Proposition 2.9 does not hold for periodic networks. However, minimizers among all possible lattices are necessarily minimizers for L^n/V among all periodic networks with the same lattice. Hence, we obtain:

Proposition 3.5. *Let N be a periodic network. If N is a minimizer among all networks with the same base graph, then N is balanced,*

$$F(v) = 0\,, \qquad \text{for all vertices } v \in V\,. \tag{3.2}$$

Proof. Invoking the variation formulas (2.2) yields the claim. □

A minimizer for L^n/V is a periodic network and so its edges have non-vanishing lengths. Therefore, the force vector F is well-defined at each vertex. Note, however, that Proposition 3.5 does not imply existence of a minimizer.

Definition 3.6. An embedded balanced network where all vertices have degree 3 is called a *Steiner network*.

In Chapter 2 we considered networks with (finitely many) boundary vertices. Theorem 2.17 stated the existence of a minimizer by estimating the number of vertices; thereby reducing the number of possible abstract graphs to a finite number. A similar approach is used here. We determine the number of vertices of the quotient network by introducing the *circuit rank* of a connected finite graph G_0:

$$\operatorname{rank} G_0 := 1 - \#\text{vertices of } G_0 + \#\text{edges of } G_0\,.$$

Note that a tree has circuit rank 0; that for any connected graph the circuit rank is a non-negative integer; and that for a d-regular graph, where all vertices have degree d, we have

$$\operatorname{rank} G_0 = 1 + \frac{d-2}{2}\#\text{vertices of } G_0\,. \tag{3.3}$$

The circuit rank is precisely the number of generators of $H_1(G_0, \mathbb{Z})$. To verify this, consider a spanning tree $T \subset G_0$ of the base graph G_0, so that $H_1(T, \mathbb{Z})$ is trivial. Reinsert the edges one by one to see that both the circuit rank of T, as well as the number of generating cycles in T, increases by 1 in each step.

We set $\operatorname{rank} N := \operatorname{rank} G_0$ for periodic networks N with base G_0 and lattice Λ. Then each generator of Λ is a lift of the image of a generator in $H_1(G_0, \mathbb{Z})$. An n-periodic network has therefore a circuit rank of at least n. The following statement serves to show that we can assume the graphs of minimizing sequences to have rank n.

Lemma 3.7. *Let $N\colon G \to \mathbb{R}^n$ be an n-periodic network with lattice $\Lambda_0 \subset \mathbb{R}^n$ and length $L(N)$. Suppose* $\operatorname{rank} G_0/\Lambda > n$*, or G_0 contains a vertex with one of the following degrees: $d = 1$, $d \geq 4$ or $d = 2$ with non-opposite edges. Then there exists an n-periodic network N' with smaller length, $L(N') < L(N)$, and the same lattice Λ_0. Moreover,* $\operatorname{rank} N' = n$ *and N' has only vertices of degree* 3 *or degree* 2 *with opposite edges.*

Proof. Consider a spanning tree G_0^T of the base graph G_0; it has circuit rank 0. We construct graphs $G_1^T \subset \ldots \subset G_n^T$ by requiring G_i^T is the union of G_{i-1}^T with an edge $e_i \in G_0 \setminus G_{i-1}^T$; the edge is chosen such that the cycles in G_i^T induce a set of lattice vectors with rank i. Now consider the (sub)network N' which is the lift of $N/\Lambda_0(G_n^T)$. By construction N' is n-periodic. So in case of rank $N > n$ the quotient network N'/Λ_0 has fewer edges than N/Λ_0 and therefore $L(N') < L(N)$.

If the assumptions on the degree hold, we will modify N'/Λ_0 in finitely many steps as in the proof of Theorem 2.17 to obtain a 3-regular network with length smaller than $L(N)$. One easily verifies that neither of those steps changes the circuit rank: We first remove all vertices of degree $d = 1$ from N', together with their incident edges. Then we split all vertices of degree $d \geq 4$ into two vertices of degree $d - 1$ and 3 repeatedly, until all vertices have degree $d \leq 3$. Finally, if the resulting graph contains a vertex of degree 2 and the network at this vertex is not balanced we can replace the two incident edges by a single edge to reduce length.

While the previous operations preserve n-periodicity they possibly do not preserve the immersion property. So assume at $N(v)$ there is more than one edge in the same direction. The initial portion of the edge then is covered at least twice. Replace it by a single edge, whose terminal may or may not be new, thereby reducing length. Iterate this construction at all vertices to define a network of shorter length such that all stars of N' become embedded. □

Remark 3.8. In Definition 3.2 of periodic networks we allow vertices of degree 2. Clearly, if a minimizer has vertices of degree 2 then the incident edges must be opposite and so the vertex can be removed. Whenever we discuss the uniqueness of minimizers we assume this is the case.

3.2 Topology of a minimizer

In this section we determine the topology of minimizers. When minimization of the length quotient is constrained to networks with a fixed lattice, the number of vertices in the base graph does not depend on the lattice chosen:

Theorem 3.9. *Among n-periodic networks with lattice $\Lambda_0 \subset \mathbb{R}^n$, there exists a network which minimizes the length quotient L^n/V. Any minimizing network $N\colon G \to \mathbb{R}^n$, after removal of vertices of degree 2 with opposite edges, is a Steiner network such that G_0 has $2n-2$ vertices.*

Proof. Consider a minimizing sequence (N_k) for the length quotient, i.e., $\lim_{k\to\infty} L(N_k) = \inf L(N)$, where the infimum is taken over n-periodic networks with lattice Λ_0.

By Lemma 3.7 we can assume that $\operatorname{rank} N_k = n$ and N_k has only vertices of degree 3. Then (3.3) implies the number of vertices is $2n-2$. There are only finitely many topological graphs with a given number of vertices. Thus by passing to a subsequence we can assume all N_k have the same topological type G. The vertices of N_k are then determined by a $(2n-2)$-tuple of points in the compact set $\mathbb{R}^n/\Lambda_0$. The edges of N_k are geodesics connecting two vertices with uniformly bounded length. This is also a compact set. Hence, for a further subsequence N_k converges to a limit N.

We claim that all edges of N attain positive length. If an edge attains length 0, a vertex with degree $d \geq 4$ results. Again by Lemma 3.7, N cannot be a minimizer of length quotient over all networks with lattice Λ_0, contradicting the fact that (N_k) is a minimizing sequence.

A similar argument shows N is embedded. An intersection point means that N can be regarded as an embedding of a length minimizing graph with a vertex of degree $d \geq 4$, contradicting again the lemma. Finally, since N is a minimizer with positive edge lengths Proposition 3.5 yields N is balanced. □

Remark 3.10. The proof indicates that our results do not change if we drop the connectivity assumption in the definition of n-periodic networks (but still require that the cycles of the underlying possibly disconnected graph span a lattice of rank n). Indeed, if a minimizer was disconnected, we could use translations to move one component as to intersect the other. Again this contradicts Lemma 3.7.

Figure 3.2: The two graphs with degree 3 on four vertices without loops: K_4 (left) and $D_1 \,\square\, D_2$ (right).

A direct consquence of Theorem 3.9 is that minimizers cannot contain loops:

Corollary 3.11. *Let N be an n-periodic network minimizing L^n/V. Then its base graph does not contain loops.*

Proof. A loop based at a vertex $v \in G$ is mapped to a straight edge in $\mathbb{R}^n$. the vertex at the lift has a pair of opposite incident edges and so contradicts balancing. □

The base graph of a 3-periodic minimizer hence must be one of the two graphs shown in Figure 3.2.

Remark. The graph K_4 denotes the complete graph on four vertices. The other graph is $D_1 \,\square\, D_2$, the Cartesian graph product of the dipole graphs D_1 and D_2 used later in the thesis. We will not make use of this combinatorial structure here.

Proposition 3.18 will state yet another constraint on the base graph of a length minimizing n-periodic network: if $n \geq 3$ then the base graph of a minimizer must be simple, i.e., it cannot contain double edges.

The number of graphs on $2n - 2$ vertices without loops where all vertices have degree 3 is finite for all $n \geq 2$. Thus for triply periodic minimizers there are two admissible base graphs. In the remainder of this chapter we will calculate the length quotient in each of these cases.

3.3 Maclaurin's inequality for elementary symmetric polynomials

In the cases we will consider, the volume V of a given network N with m labelled edges of length $(x_1, \ldots, x_m)$ is a polynomial $P(x_1, \ldots, x_m)$. Thus the task to minimize the quotient L^3/V is equivalent to maximizing the polynomial P under the length constraint $L = 1$, where $L(x_1, \ldots, x_m) := x_1 + \ldots + x_m$.

In the most symmetric case, P is the *elementary symmetric polynomial* of degree k,

$$P_k(x_1, \ldots, x_m) := \sum_{1 \leq j_1 < \ldots < j_k \leq m} x_{j_1} \cdots x_{j_k}\,, \qquad 1 \leq k \leq m\,.$$

We also set $P_0(x_1, \ldots, x_m) := 1$. We can estimate these polynomials by the length:

Lemma 3.12 (Maclaurin's inequality)**.** *If $x_i \geq 0$ for all $1 \leq i \leq m$ and $k \geq 2$ then*

$$P_k(x_1, \ldots, x_m) \leq \binom{m}{k} \left(\frac{x_1 + \ldots + x_m}{m} \right)^k, \tag{3.4}$$

where equality holds if and only if $x_1 = \ldots = x_m$.

In particular, for degree $k \geq 2$ the elementary symmetric polynomial P_k takes its maximum under the length constraint $L = 1$ exactly at $(\frac{1}{m}, \ldots, \frac{1}{m})$. One way to prove Maclaurin's inequality is to use Newton's inequality, see [HLP52]. We present a more direct proof here, inspired by our application.

Proof. We prove (3.4) by induction over m. The base case is $m = k$, where $P_k = x_1 \cdots x_m$. Then (3.4) is the estimate on geometric and arithmetic mean.

For the step suppose $m > k \geq 2$. We claim (3.4) holds strictly if some but not all x_i vanish. In view of the symmetry of (3.4) we may assume $x_m = 0$.

Note that $P_k(x_1,\ldots,x_{m-1},0)$ is an elementary symmetric polynomial of degree k in $m-1$ variables, and so the induction hypothesis gives

$$P_k(x_1,\ldots,x_{m-1},0) \le \binom{m-1}{k}\Big(\frac{x_1+\ldots+x_{m-1}}{m-1}\Big)^k .$$

We estimate the right hand side, using the strict inequality

$$\begin{aligned}\binom{m-1}{k}\frac{1}{(m-1)^k} &= \frac{1}{k!}\,\frac{m-1}{m-1}\cdots\frac{m-k}{m-1}\\ &< \frac{1}{k!}\,\frac{m}{m}\cdots\frac{m-k+1}{m}\\ &= \binom{m}{k}\frac{1}{m^k}\,.\end{aligned}$$

If not all x_i vanish this yields strict inequality in (3.4), as claimed.

Since (3.4) is scaling invariant, it is sufficient to prove this inequality under the length constraint $L=1$. The continuous function P_k attains a maximum over the compact set $L^{-1}(1)\subset[0,\infty)^m$ at some point $z=(z_1,\ldots,z_m)$. Note that $z\neq 0$, that we have equality in (3.4) for $x=(\frac{1}{m},\ldots,\frac{1}{m})$, and that the induction hypothesis, in form of the claim, gives strict inequality in (3.4) on $\partial([0,\infty)^m)\setminus\{0\}$. Thus z must be an interior point of $[0,\infty)^m$. Since z is critical for P_k under the smooth constraint $L=1$ we obtain the necessary condition

$$\nabla P_k(z) = \lambda\nabla L(z) = \lambda(1,\ldots,1) \tag{3.5}$$

with $\lambda\in\mathbb{R}$ a Lagrange multiplier. It remains to show this implies $z_1 = \ldots = z_m$. Then since z assigns equality to (3.4) and z was chosen maximally, the proof of the induction step is completed.

Since P_k is elementary symmetric, for $i\neq j$ we can express P_k at any point $x=(x_1,\ldots,x_m)$ as

$$P_k(x) = x_ix_jQ_0(x) + (x_i+x_j)Q_1(x) + Q_2(x)\,,$$

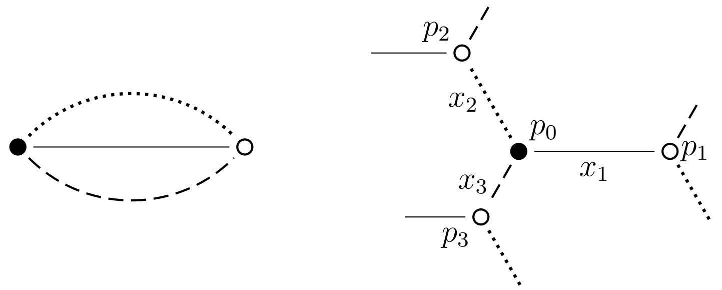


Figure 3.3: The dipole graph D_3 and a network covering it.

where Q_0, Q_1, Q_2 are polynomials in $m-2$ variables, independent of x_i and x_j. From (3.5) we conclude $\partial_i P_k(z) = \partial_j P_k(z)$ for all $1 \le i, j \le m$, so that

$$z_j Q_0(z) + Q_1(z) = z_i Q_0(z) + Q_1(z)\,.$$

Moreover, since $k \ge 2$ and $z_i > 0$ for all i, the polynomial Q_0 cannot vanish at z. Thus indeed $z_i = z_j$ for all i, j. □

3.4 Doubly periodic Steiner networks

We find it instructive to present the case of dimension $n = 2$ before studying the more involved case $n = 3$. We first determine the topology of the quotient graph of a minimizer for prescribed lattice. By Theorem 3.9 it has 2 vertices, and by Corollary 3.11 it has no loops. The only connected 3-regular graph on 2 vertices without loops is D_3, see Figure 3.3. Hence we obtain:

Lemma 3.13. *A doubly periodic network $N \subset \mathbb{R}^2$ minimizing the length quotient L^2/A for prescribed lattice Λ is Steiner on 2 vertices with the dipole graph D_3 as a quotient.*

Since the edges of a Steiner network enclose 120°-angles, a minimizing network can be described in terms of three edge lengths alone:

Lemma 3.14. *Up to isometries of $\mathbb{R}^2$, a doubly periodic Steiner network $N \subset \mathbb{R}^2$ with quotient D_3 is uniquely determined by its three edge* *lengths* $x_1, x_2, x_3 > 0$. *Its length and spanned area are*

$$L = x_1 + x_2 + x_3 \qquad \textit{and} \qquad A = \frac{\sqrt{3}}{2}(x_1x_2 + x_1x_3 + x_2x_3)\,.$$

Proof. The two vertices of D_3 correspond to a vertex $p_0 \in N$ and the incident vertices $p_1, p_2, p_3 \in N$, where the labelling relates to the lengths as in Figure 3.3. Then the lattice Λ of N is spanned, for instance, by $g_1 := p_1 - p_3$ and $g_2 := p_2 - p_3$. Specifically, we may assume that up to isometry we have

$$p_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad p_1 = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad p_2 = x_2 \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad p_3 = x_3 \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix},$$

and so the lattice Λ has area

$$A = |\det(g_1, g_2)| = \frac{\sqrt{3}}{2}(x_1x_2 + x_1x_3 + x_2x_3)\,. \qquad \square$$

As might be expected, the optimal doubly periodic network is given by the tessellation of $\mathbb{R}^2$ with regular hexagons:

Proposition 3.15. *For each doubly periodic network* $N \subset \mathbb{R}^2$ *we have*

$$\frac{L^2}{A} \geq 2\sqrt{3}\,. \tag{3.6}$$

Equality holds if N *has the quotient* D_3 *and the three edge lengths of* N *are equal; then the lattice is hexagonal.*

Proof. For a prescribed lattice Λ, Lemma 3.13 asserts the existence of a Steiner network N_0 with quotient D_3 which is a minimizer, $(L^2/A)(N) \geq (L^2/A)(N_0)$, where the inequality is strict unless N has quotient D_3. According to Lemma 3.14, the edge lengths $x_1, x_2, x_3 > 0$ determine N_0, and the area $A(N_0)$ is a multiple of the elementary symmetric polynomial of degree 2 in three variables. Thus Maclaurin's inequality (3.4) implies (3.6) for N_0:

$$\begin{aligned} A(N_0) &= \frac{\sqrt{3}}{2}(x_1x_2 + x_1x_3 + x_2x_3) \\ &\le \frac{3\sqrt{3}}{2}\left(\frac{x_1+x_2+x_3}{3}\right)^2 \\ &= \frac{1}{2\sqrt{3}}(L(N_0))^2 \end{aligned}$$

In particular, (3.6) follows for N. To discuss the equality case, note that for N with quotient D_3 and $x_1 = x_2 = x_3$, equality in (3.6) is obvious. But by the above and Lemma 3.12 the equality can only hold for this case. $\square$

3.5 Triply periodic Steiner networks covering $D_1 \square D_2$

Our approach to triply periodic Steiner networks is similar to the doubly periodic case. However, as pointed out in the beginning of this chapter, Theorem 3.9 and Lemma 3.7 allow exactly two distinct topologies of minimizing Steiner networks:

Lemma 3.16. *The topological graph of a triply periodic Steiner network, minimizing length for a prescribed lattice Λ, is K_4 or $D_1 \square D_2$.*

We analyze the case $D_1 \square D_2$ first since our analysis of the more prominent K_4-case makes use of it. In both cases we can parameterize the space of networks by the edge lengths $x_1, \dots, x_6$ of the six edges $e_1, \dots, e_6$ in the quotient N/Λ; for $D_1 \square D_2$ there is a further angle parameter. This will follow from considering the tangent planes at the vertices; note that the Steiner condition implies that each vertex is coplanar with its three neighbours. In dimension $n = 3$, the angles between the different tangent planes turn out to be independent of the edge lengths.

With respect to a labelling as in Figure 3.4 we state:

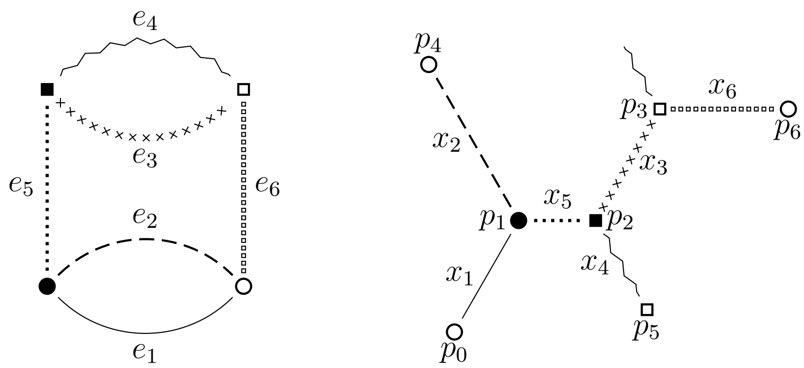


Figure 3.4: The graph $D_1 \square D_2$ and its covering network in schematic view.

Lemma 3.17. *Let $N \subset \mathbb{R}^3$ be a triply periodic Steiner network with quotient $D_1 \square D_2$. Then, up to isometry, the network N is uniquely determined by its six edge lengths $x_1, \ldots, x_6 > 0$ and an angle $\alpha \in (0, \pi)$. Moreover, N has length $L = \sum_i x_i$ and, for a labelling of the edge lengths as in Figure 3.4, the spanned volume is*

$$V = \frac{3}{4}\sin(\alpha)\Big(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \\ + (x_5 + x_6)(x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4)\Big)\,. \tag{3.7}$$

We will see that all edges of N are contained in two sets of parallel planes which make an angle α to be chosen independently of the edge lengths. The limiting cases $\alpha = 0$ and π relate to a doubly periodic network.

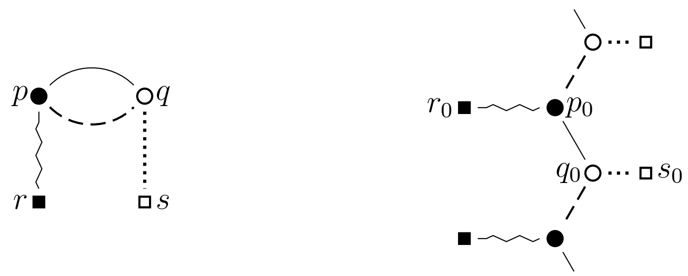


Figure 3.5: A Steiner network with double edges. The stars of the two doubly connected vertices lie in a common plane.

Proof. Consider a connected subgraph $\tilde{N} \subset N$ with vertices $p_0, \ldots, p_6$ as in Figure 3.4 such that p_0, p_4, p_6, as well as p_3, p_5 are identified in the quotient. We may assume p_1 is the orgin, p_2 is on the x-axis, and p_0, p_4 are in the xy-plane. Balancing then implies

$$p_0 = x_1 \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = x_5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_4 = x_2 \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}. \quad (3.8)$$

The tangent plane at p_2 must be a rotation about the x-axis of the tangent plane at p_1 by an angle $\alpha \in [0, 2\pi)$. Let $A_\alpha \in SO(3)$ denote such a rotation. The Steiner condition then implies that $p_3 - p_2$ points in the same direction as $p_1 - p_0$ rotated by A_α. The same applies to $p_5 - p_2$ and $p_1 - p_4$. That is,

$$\begin{aligned} p_3 &= p_2 + x_3\, A_\alpha \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_3 + 2x_5 \\ x_3\sqrt{3}\cos\alpha \\ x_3\sqrt{3}\sin\alpha \end{pmatrix}, \\ p_5 &= p_2 + x_4\, A_\alpha \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_4 + 2x_5 \\ -x_4\sqrt{3}\cos\alpha \\ -x_4\sqrt{3}\sin\alpha \end{pmatrix}. \end{aligned} \quad (3.9)$$

Triple periodicity implies $\alpha \neq 0 \bmod \pi$ and changing α to $\alpha \pm \pi$ corresponds to a change of numbering of the vertices p_3 and p_5. So we may assume $\alpha \in (0, \pi)$. Using the Steiner condition we see that for a pair of vertices which are doubly connected in N/Λ the normals must agree. Since p_6 and p_0 are identified in N/Λ the vector $p_6 - p_3$ points in the same direction as $p_2 - p_1$. So we have

$$p_6 = p_3 + x_6 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_3 + 2x_5 + 2x_6 \\ x_3\sqrt{3}\cos\alpha \\ x_3\sqrt{3}\sin\alpha \end{pmatrix}. \quad (3.10)$$

The three vectors

$$g_1 := p_4 - p_0 = \frac{1}{2}\begin{pmatrix} x_1 - x_2 \\ \sqrt{3}(x_1 + x_2) \\ 0 \end{pmatrix},$$

$$g_2 := p_5 - p_3 = \frac{1}{2}\begin{pmatrix} x_4 - x_3 \\ -\sqrt{3}(x_3 + x_4)\cos\alpha \\ -\sqrt{3}(x_3 + x_4)\sin\alpha \end{pmatrix}, \tag{3.11}$$

$$g_3 := p_6 - p_0 = \frac{1}{2}\begin{pmatrix} x_1 + x_3 + 2x_5 + 2x_6 \\ \sqrt{3}(x_1 + x_3\cos\alpha) \\ x_3\sqrt{3}\sin\alpha \end{pmatrix}$$

span the lattice Λ; indeed, an inspection of Figure 3.4 shows they correspond to minimal cycles in the abstract graph. Then $|\det(g_1, g_2, g_3)|$ can be computed as (3.7). □

As an aside, we use the reasoning of Lemma 3.17 to show that a minimizer $N \subset \mathbb{R}^n$ of L^n/V for prescribed lattice Λ cannot contain double edges for $n \geq 3$. According to Theorem 3.9 the network N is an n-periodic Steiner network. Let $p_0, q_0 \in N$ be two adjacent vertices, and suppose they project onto doubly connected vertices $p, q \in N/\Lambda$. Denote by r_0 the neighbour of p_0 which does not project to q, and by s_0 the neighbour of q not projecting to p, see Figure 3.5. Then the Steiner condition shows the vectors $r_0 - p_0$ and $s_0 - q_0$ are parallel and point into opposite directions. Now move p_0 and q_0 simultanously in one of these directions: For $0 \leq t < 1$, replace p_0 by $p_0^t := p_0 + t(r_0 - p_0)$ and q_0 by $q_0^t := q_0 + t(r_0 - p_0)$, and similarly so for all other lifts of p, q. We obtain an n-periodic Steiner network N^t with the same lattice Λ and $L(N_t) = L(N)$. The limiting network N_1 with lattice Λ has length $L(N_1) = L(N)$ and so is again minimizing. However, N_1 has one vertex of degree 4, thereby contradicting Lemma 3.7. Our reasoning proves:

Proposition 3.18. *An n-periodic minimizer of L^n/V with $n \geq 3$ covers a* *simple graph on* $2n-2$ *vertices of degree* 3.

Remark. The number of connected 3-regular simple graphs on $2n-2$ vertices, i.e., cubic graphs, is rapidly growing in $n \geq 3$, see oeis.org.

The proposition implies that a triply periodic minimizer can only have the quotient K_4. Thus if we are merely interested in establishing Theorem 3.1 it may appear that we do not need the estimate for the quotient $D_1 \,\square\, D_2$, stated in the next theorem. However, a limiting case of (3.12) below will enter the proof of Theorem 3.23, and the equality result will also be used in Section 3.7.

To determine optimal networks with quotient $D_1 \,\square\, D_2$ we now solve a standard calculus problem, namely we maximize the function V under a constraint for L. Interestingly enough, up to similarity of $\mathbb{R}^3$ there is a one-parameter family of optimal networks, meaning that these networks are not strictly stable:

Theorem 3.19. *Let* $N \subset \mathbb{R}^3$ *be a triply periodic Steiner network with quotient* $D_1 \,\square\, D_2$. *Then*

$$\frac{L^3}{V} \geq \frac{81}{4}\,, \tag{3.12}$$

where equality holds if and only if

$$x_1 = x_2 = x_3 = x_4 = 2x_5 + 2x_6 \quad \textit{and} \quad \alpha = \frac{\pi}{2}\,. \tag{3.13}$$

In the equality case the lattice is generated, up to similarity, by $(0,1,0)$, $(0,0,1)$, $\frac{1}{2}(\sqrt{3},1,1)$.

Proof. Admitting vanishing edge lengths, we will show the inequality in a form implying (3.12), namely

$$V \leq \frac{4}{81}L^3 \quad \text{for all } x \in [0,\infty)^6 \text{ and } \alpha \in (0,\pi)\,, \tag{3.14}$$

with equality precisely for (3.13).

For fixed $x = (x_1, \dots, x_6)$ clearly L is independent of α, while (3.7) gives that V is maximal exactly at $\alpha = \pi/2$. Moreover, both V and L depend on x_5, x_6 only through $y := x_5 + x_6$. Thus in order to establish (3.14) we may fix α to $\pi/2$ and consider the functions induced by L and V on the domain $[0,\infty)^5 \ni (x_1, x_2, x_3, x_4, y)$. For the remainder of the proof we denote these continuous functions again by L and V.

We claim that (3.14) holds along the boundary of $[0,\infty)^5$. Trivially, this is true at 0. Otherwise let (x_1, x_2, x_3, x_4, y) be a point where at least one coordinate vanishes. In case $y = 0$ the volume is

$$V = \frac{3}{4}\Big(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4\Big)\,.$$

The right-hand side contains the elementary symmetric polynomial of degree $k = 3$ in $m = 4$ variables and so indeed, by Maclaurin's inequality (3.4),

$$V \le 3\Big(\frac{x_1 + x_2 + x_3 + x_4}{4}\Big)^3 = \frac{3}{64}L^3 < \frac{4}{81}L^3\,.$$

The other case is that some x_i vanishes for $i = 1, 2, 3$, or 4. In view of the symmetry of V and L it suffices to consider the case $x_1 = 0$. Under this assumption Maclaurin's inequality gives

$$V = \frac{3}{4}x_2(x_3x_4 + x_3y + x_4y) \le \frac{1}{4}x_2(x_3 + x_4 + y)^2\,.$$

Then the claim follows from the estimate on geometric and arithmetic mean,

$$V \le x_2\,\frac{x_3 + x_4 + y}{2}\,\frac{x_3 + x_4 + y}{2} \le \frac{1}{27}(x_2 + x_3 + x_4 + y)^3 = \frac{1}{27}L^3 < \frac{4}{81}L^3\,.$$

We now proceed as in the proof of Maclaurin's inequality. The continuous function V attains its maximum on the compact set $L^{-1}(1) \subset [0,\infty)^5 \setminus \{0\}$ at some point $z := (x_1, \dots, x_4, y)$.

One easily verifies $V = 4\,L^3/81$ if (3.13) holds. Thus our claim implies that in fact $z \in (0,\infty)^5$. For the set $(0,\infty)^5$, the point z is critical for V

under the constraint $L = 1$, and so

$$\nabla V(z) = \lambda \nabla L(z)\,,$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Equivalently,

$$\frac{3}{4}\begin{pmatrix} x_2x_3 + x_2x_4 + x_3x_4 + x_3y + x_4y \\ x_1x_3 + x_1x_4 + x_3x_4 + x_3y + x_4y \\ x_1x_2 + x_1x_4 + x_2x_4 + x_1y + x_2y \\ x_1x_2 + x_1x_3 + x_2x_3 + x_1y + x_2y \\ x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{3.15}$$

We claim this implies $x_1 = x_2 = x_3 = x_4 = 2y$. For the proof, we consider dot products of (3.15) with four different vectors. Namely, the product with $(1, -1, 0, 0, 0)$ gives $(x_2 - x_1)(x_3 + x_4) = 0$, the product with $(0, 0, 1, -1, 0)$ gives $(x_1 + x_2)(x_4 - x_3) = 0$. Moreover, for $(0, 1, 0, -1, 0)$ we obtain $(x_4 - x_1)(x_1 + x_4 + 2y) = 0$, and for $(0, 1, 0, 0, -1)$ we obtain $x_1(2y - x_1) = 0$. Since z has positive coordinates our four equations prove the claim.

We have shown there is a unique critical point $z \in (0, \infty)^5$ for V under the constraint $L = 1$, where V attains its maximal value $V(z) = 4/81$. This implies the inequality (3.14) first for $L = 1$, and then, by the scaling invariance of L^3/V, in general. Finally, the uniqueness of z implies that in general equality holds if and only if (3.13) holds; to verify the lattice vectors use (3.11). $\square$

3.6 The srs network covering the K_4 graph

We discuss the network related to the gyroid. Each vertex of a Steiner network has a well-defined affine tangent plane, containing the edge vectors to the incident vertices; each vertex in N/Λ defines a tangent plane up to translation. (We avoid the usage of normal vectors since the tangent planes are unoriented.)

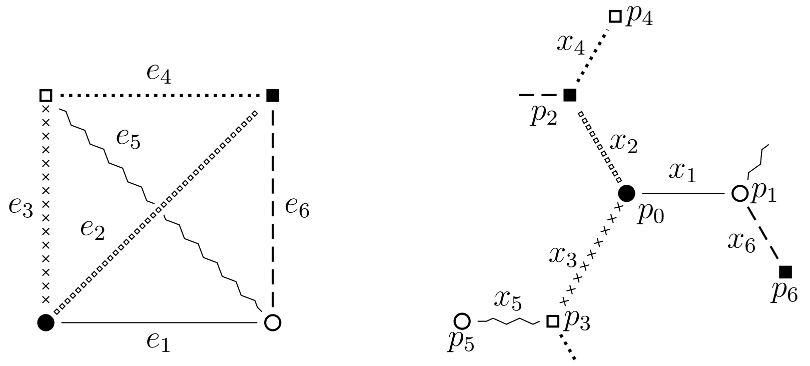


Figure 3.6: The graph K_4 and the labelling of the network covering it.

For a Steiner network with quotient K_4 we use balancing and the fact that each pair of vertices in K_4 is connected with an edge to show that the four tangent planes are perpendicular to the four space diagonal directions:

Lemma 3.20. *Let $N \subset \mathbb{R}^3$ be a triply periodic Steiner network with quotient graph K_4. Then the four tangent planes of N/Λ are parallel to the four faces of a regular tetrahedron. Consequently, up to isometry of $\mathbb{R}^3$, the network N is uniquely defined by its six edge lengths $x_1, \ldots, x_6 > 0$.*

Proof. From N we pick a connected subgraph which contains a vertex p_0 and its three neighbours p_1, p_2, p_3, representing the vertices of N/Λ. Without loss of generality we may assume p_0 to be the origin, p_1 to lie on the x-axis and p_2, p_3 to lie in the xy-plane. That is, we assume

$$p_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = x_2 \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad p_3 = x_3 \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \tag{3.16}$$

where $x_i > 0$ is the edge length of the edge incident to p_i.

Let $p_6 \neq p_0$ be a vertex incident to p_1, compare Figure 3.6. Copying the reasoning of the proof of Lemma 3.17 we find, in terms of some rotation

A_β about the x-axis, where $-\pi < \beta < \pi$:

$$p_6 - p_1 = \frac{x_6}{x_2} A_\beta (p_0 - p_2) = x_6 A_\beta \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} = x_6 \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2}\cos(\beta) \\ -\frac{\sqrt{3}}{2}\sin(\beta) \end{pmatrix}.$$

Then $\min\left\{|\beta|, \pi - |\beta|\right\}$ represents the dihedral angle between the two tangent planes at p_0 and p_1.

In the quotient N/Λ, the vertex p_6 must be identified with one of the four vertices $p_0, \ldots, p_3$. Since the shortest cycle in K_4 consists of three edges this vertex must be either p_2 or p_3. Suppose p_6 is identified with p_2. The tangent planes at these two points are parallel. Hence the balancing equation (3.2) implies that the vectors $p_2 - p_0$ and $p_6 - p_1$ enclose 120 degrees, and the sum of the two unit vectors pointing into these directions must be a unit vector:

$$1 = \left| \frac{p_0 - p_2}{|p_0 - p_2|} + \frac{p_1 - p_6}{|p_1 - p_6|} \right| = \left| \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2}(-1 + \cos(\beta)) \\ \frac{\sqrt{3}}{2}\sin(\beta) \end{pmatrix} \right| = \sqrt{\frac{3}{2}} \sqrt{1 - \cos(\beta)}\,.$$

The other case is that p_6 is identified with p_3. Then, similarly,

$$1 = \left| \frac{p_1 - p_6}{|p_1 - p_6|} + \frac{p_0 - p_3}{|p_0 - p_3|} \right| = \left| \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2}(1 + \cos(\beta)) \\ \frac{\sqrt{3}}{2}\sin(\beta) \end{pmatrix} \right| = \sqrt{\frac{3}{2}} \sqrt{1 + \cos(\beta)}\,.$$

From both cases we conclude $|\cos(\beta)| = 1/3$, and so the dihedral angle of the tangent planes at p_0 and p_1 is the tetrahedral angle $\arccos(1/3) \approx 70.53°$.

In K_4, any pair of vertices is connected by an edge, and so the same argument applies to any pair of vertices p_i, p_j of N/Λ. But four planes in $\mathbb{R}^3$ can only have pairwise dihedral angles $\arccos(1/3)$ if they are parallel to the faces of a regular tetrahedron.

Finally, lengths and tangent planes determine a Steiner network completely up to isometry. $\square$

For the next statement we choose to label the six edges $e_1, \ldots, e_6$, such that the edges e_i and e_{i+3} do not have endpoints in common, see Figure 3.6. We let x_i be the length of e_i.

Lemma 3.21. *Let N be a triply periodic Steiner network with quotient K_4. Then N has length $L = \sum_i x_i$ and the spanned volume is*

$$\begin{aligned} V = \frac{1}{\sqrt{2}}\Big(&x_1x_2x_4 + x_1x_2x_5 + x_1x_2x_6 + x_1x_3x_4 \\ &+ x_1x_3x_5 + x_1x_3x_6 + x_1x_4x_5 + x_1x_4x_6 + x_2x_3x_4 + x_2x_3x_5 \\ &+ x_2x_3x_6 + x_2x_4x_5 + x_2x_5x_6 + x_3x_4x_6 + x_3x_5x_6 + x_4x_5x_6\Big)\,. \end{aligned} \tag{3.17}$$

The sum extends over all possible products of three edge lengths except for those relating to triples of concurrent edges.

Remark 3.22. By Lemma 3.20, lengths and tangent planes determine a Steiner network completely up to isometry. Up to rigid motion, however, there are two different Steiner networks covering K_4 with the same edge lengths. The isometry mapping the two networks onto another is a reflection which corresponds to a sign change of β. Note that the four tangent planes at the vertices of a network are the tangent planes of a regular tetrahedron. Hence, the choice of any two tangent planes determines the other two.

Proof. After isometry of $\mathbb{R}^3$ we may assume the coordinates are as in (3.16). For $i = 1, 2, 3$ let $A^i_\beta \in SO(3)$ be the rotation fixing p_i with an angle $\beta = \arccos(1/3)$. In view of Remark 3.22, possibly by replacing β by $-\beta$, the three vectors

$$g_1 := (p_0 - p_2) + (p_1 - p_0) + \frac{x_6}{x_2} A_\beta^1 (p_0 - p_2) = \begin{pmatrix} x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_6 \\ -\frac{\sqrt{3}}{2}x_2 - \frac{1}{2\sqrt{3}}x_6 \\ -\sqrt{\frac{2}{3}}\, x_6 \end{pmatrix},$$

$$g_2 := (p_0 - p_3) + (p_2 - p_0) + \frac{x_4}{x_3} A_\beta^2 (p_0 - p_3) = \begin{pmatrix} -\frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{\sqrt{3}}{2}x_2 + \frac{\sqrt{3}}{2}x_3 + \frac{1}{\sqrt{3}}x_4 \\ -\sqrt{\frac{2}{3}}\, x_4 \end{pmatrix},$$

$$g_3 := (p_0 - p_1) + (p_3 - p_0) + \frac{x_5}{x_1} A_\beta^3 (p_0 - p_1) = \begin{pmatrix} -x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_5 \\ -\frac{\sqrt{3}}{2}x_3 - \frac{1}{2\sqrt{3}}x_5 \\ -\sqrt{\frac{2}{3}}\, x_5 \end{pmatrix}$$

are linearly independent and span the lattice Λ. To verify (3.17), calculate

$$\det(g_1, g_2, g_3) = \frac{-1}{2\sqrt{2}} \det \begin{pmatrix} 2x_1 + x_2 + x_6 & -x_2 + x_3 & -2x_1 - x_3 - x_5 \\ -x_2 - \frac{1}{3}x_6 & x_2 + x_3 + \frac{2}{3}x_4 & -x_3 - \frac{1}{3}x_5 \\ x_6 & x_4 & x_5 \end{pmatrix}.$$

□

Theorem 3.23. *Let N be a triply periodic Steiner network in $\mathbb{R}^3$ with quotient K_4. Then*

$$\frac{L^3}{V} \geq \frac{27}{\sqrt{2}} \approx 19.09\,,$$

where equality holds if and only if all edge lengths of N coincide and the lattice is body-centered cubic.

Proof. We follow the strategy of the proof of Theorem 3.19. For the present case, L and V are functions of the six edge lengths, see Lemma 3.21.

We first verify the strict inequality $L^3 > (27/\sqrt{2})V$ along the boundary of $[0,\infty)^6$ without the point 0. Assume that at least one x_i vanishes. By symmetry of V and L in all variables we may assume $x_6 = 0$. Then the volume V becomes

$$V = \frac{1}{\sqrt{2}}\big(x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5\big)\,.$$

This expression matches the volume (3.7) of a `ths` network with $x_6 = 0$ and $\alpha = \arccos(1/3) = \arcsin(2\sqrt{2}/3)$, after exchanging x_3 and x_5. Using (3.12), this proves, as desired

$$L^3 \geq \frac{81}{4}V > \frac{27}{\sqrt{2}}V\,.$$

Thus it suffices to minimize L^3/V over the set where all coordinates are strictly positive. We maximize V under the constraint $L = 1$. A critical point $z = (x_1, \dots, x_6) \neq 0$ satisfies

$$\nabla V(z) = \lambda \nabla L(z)\,,$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. By (3.17) this is equivalent to

$$\frac{1}{\sqrt{2}}\begin{pmatrix} x_2x_4 + x_3x_4 + x_2x_5 + x_3x_5 + x_4x_5 + x_2x_6 + x_3x_6 + x_4x_6 \\ x_1x_4 + x_3x_4 + x_1x_5 + x_3x_5 + x_4x_5 + x_1x_6 + x_3x_6 + x_5x_6 \\ x_1x_4 + x_2x_4 + x_1x_5 + x_2x_5 + x_1x_6 + x_2x_6 + x_4x_6 + x_5x_6 \\ x_1x_2 + x_1x_3 + x_2x_3 + x_1x_5 + x_2x_5 + x_1x_6 + x_3x_6 + x_5x_6 \\ x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_2x_6 + x_3x_6 + x_4x_6 \\ x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_3x_4 + x_2x_5 + x_3x_5 + x_4x_5 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\,. \tag{3.18}$$

We claim this implies z satisfies $x_1 = \ldots = x_6 = \frac{1}{6}$. Again we compute dot products of vectors with (3.18). For $(1, 0, -1, 1, 0, -1)$ we obtain $-2x_1x_4 + 2x_3x_6 = 0$, and $(0, 1, -1, 0, 1, -1)$ gives $-2x_2x_5 + 2x_3x_6 = 0$. Equivalently, $x_1x_4 = x_2x_5 = x_3x_6$ or, since none of the coordinates can vanish, $x_4 =$

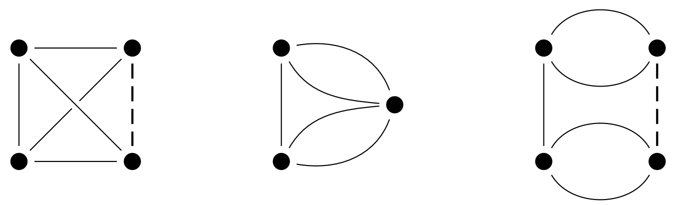


Figure 3.7: The graph shown in the middle arises as a limit of the graph K_4 (left) or of $D_1 \square D_2$ (right) when the dashed edge is contracted.

x_3x_6/x_1 and $x_5 = x_3x_6/x_2$. Using this, we conclude

$$\begin{aligned}
0 &= \langle \nabla V(z), (1,-1,0,0,0,0)\rangle = (x_2 - x_1)\frac{x_6(x_1x_2 + x_1x_3 + x_2x_3 + x_3x_6)}{x_1x_2}\,,\\
0 &= \langle \nabla V(z), (0,0,1,-1,0,0)\rangle = (x_6 - x_1)\frac{x_1x_2 + x_1x_3 + x_2x_3 + x_3x_6}{x_1}\,,\\
0 &= \langle \nabla V(z), (0,0,0,0,1,-1)\rangle = (x_2 - x_3)\frac{x_6(x_1x_2 + x_1x_3 + x_2x_3 + x_3x_6)}{x_1x_2}\,.
\end{aligned}$$

Therefore $x_1 = x_2 = x_3 = x_6$ and, using $x_1x_4 = x_2x_5 = x_3x_6$, these must agree with $x_4 = x_5$. This proves the claim. Reasoning literally as in the proof of Theorem 3.19 concludes the proof. $\square$

Remark. The proof of Theorem 3.23 asserts that if $x_6 = 0$ the length and volume of the srs network and the ths network with $\alpha = \arccos(1/3)$ agree. In particular, the topological graphs of the networks agree, see Figure 3.7.

We would like to draw another consequence of Lemma 3.20.

Proposition 3.24. *For each choice of edge lengths* $x_1, \dots, x_6 > 0$ *there exists a Steiner network in* $\mathbb{R}^3$ *with quotient* K_4*. Up to isometry, its vertices* $p_1, \dots, p_6$ *are uniquely given by* (3.16) *as well as*

$$p_4 = p_2 + x_4\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}, \quad p_5 = p_3 + x_5\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}, \quad p_6 = p_1 + x_6\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix},$$

and the lattice is $\Lambda = (p_6 - p_2)\mathbb{Z} + (p_4 - p_3)\mathbb{Z} + (p_5 - p_1)\mathbb{Z}$*.*

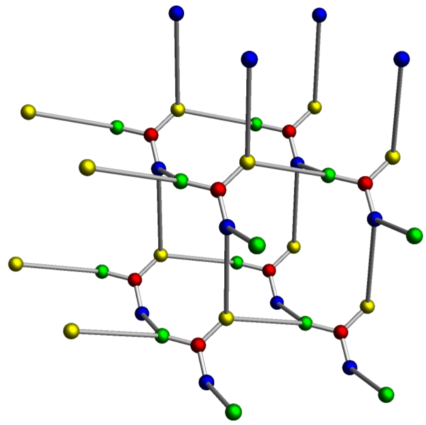


Figure 3.8: A triply periodic Steiner network with quotient K_4 where the lattice is primitive. The eight vertices shown in red correspond to the eight vertices of a cube.

Setting, for instance, $3x_1 = 3x_2 = 3x_3 = x_4 = x_5 = x_6 = 3$ gives

$$g_1 = \begin{pmatrix} 3 \\ -\sqrt{3} \\ -\sqrt{6} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ 2\sqrt{3} \\ -\sqrt{6} \end{pmatrix}, \quad g_3 = \begin{pmatrix} -3 \\ -\sqrt{3} \\ -\sqrt{6} \end{pmatrix}.$$

These vectors have the same length and are orthogonal so that the lattice is primitive cubic. Moreover, $L^3/V = 16\sqrt{2} \approx 22.63$. See Figure 3.8.

3.7 Homotopy from a ths-network to a K_4-network which decreases length

We know from Theorem 3.19 that minimizing networks with quotient $D_1 \square D_2$ are part of the one-parameter family (3.13) with a fixed lattice. In the present section we show there is a continuous 1-parameter family leading from a given minimizing ths network to a network with smaller length and quotient K_4.

The transition between the two distinct topological types is achieved via a network which has one degree-4 vertex in the quotient. There are two

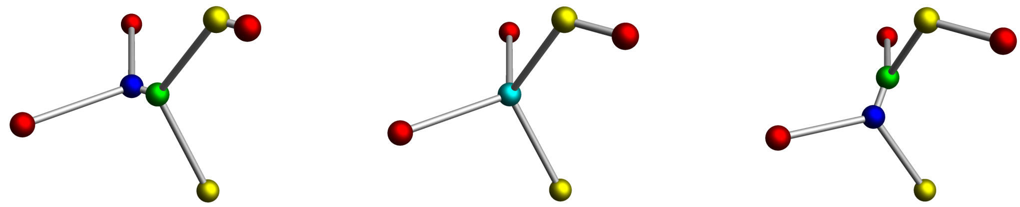


Figure 3.9: Homotopy of Theorem 3.25, schematically. A ths network (left) can continuously be deformed into a triply periodic network covering K_4 (right). The homotopy preserves the lattice but decreases length. The change of topology occurs when two vertices coincide (center).

ways to split this vertex into two degree-3 vertices, as is well-known from the Steiner tree problem on four vertices. See Figures 3.7 and 3.10 for the topological picture, and Figure 3.9 for the geometry.

Theorem 3.25. *Let N_{-1} be a minimizing* ths *network as in* (3.13)*, scaled such that $x_1 = 1$, and with lattice Λ_0. Then there is a continuous family of triply periodic networks $N_t \subset \mathbb{R}^3$ for $t \in [-1, 1]$ from N_{-1} to a Steiner network N_1 with the following properties:*

- *All N_t have the same lattice Λ_0 and so the same volume V.*
- *N_t is a network with quotient graph $D_1 \,\square\, D_2$ for $-1 \le t < 0$, and K_4 for $0 < t \le 1$.*
- *The length $t \mapsto L(N_t)$ is non-increasing and $L(N_1) < L(N_{-1})$.*

We will specify the networks N_t in terms of six generating vertices p_i^t for $i = 1, \ldots, 6$, as well as the straight segments joining pairs of these vertices given by Figure 3.10. The lattice Λ_0 then generates N_t.

Let us first describe the networks N_t for negative t in terms of the ths family (3.13). The given ths network N_{-1}, subject to (3.13) with $x_1 = 1$, has length $L(N_{-1}) = 9/2$ and according to (3.11) its lattice Λ_0 is generated by

$$g_1 = \left(0, \sqrt{3}, 0\right), \quad g_2 = \left(0, 0, -\sqrt{3}\right), \quad g_3 = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right).$$

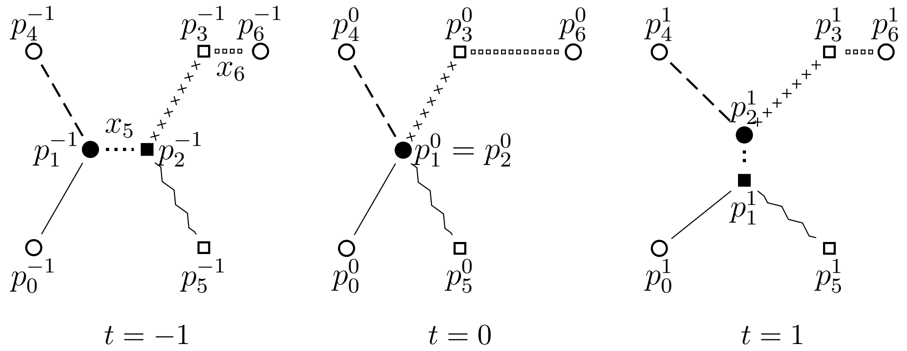


Figure 3.10: Homotopy of Theorem 3.25, schematically. The transition from the graph $D_1 \square D_2$ (left) to the K_4 graph (right) is via the graph (center) with a vertex of degree 4.

The network N_{-1} is uniquely determined by the edge length $x_5 =: \xi \in (0, \frac{1}{2})$. For $-1 \le t < 0$ we define N_t as the ths network, subject to (3.13), still with $x_1 = 1$ but with $x_5 := |t|\xi$. This prescribes the six vertices p_i^t for $t \in [-1, 0)$ by (3.8) to (3.10).

For $t = 0$ the limiting data

$$x_1 = x_2 = x_3 = x_4 = 1\,, \quad x_5 = 0\,, \quad x_6 = \frac{1}{2}\,, \quad \alpha = \frac{\pi}{2}$$

similarly defines a network N_0 with $p_1^0 = p_2^0 = 0$. Inspection of Figure 3.10 shows that under this conditionthe network N_0 can also be understood as a limit of networks with quotient K_4, where again the edge between the points p_1^t and p_2^t has length tending to 0 as $t \searrow 0$.

To complete the proof of Theorem 3.25, let us now make the deformation of the network N_0 into a Steiner network with quotient K_4 explicit.

Lemma 3.26. *There is a continuous family N_t, $0 \le t \le 1$, of networks with lattice Λ_0, such that N_1 is a Steiner network, the length $t \mapsto L(N_t)$ is (strictly) decreasing, and for $0 < t \le 1$ the quotient graph is K_4.*

Proof. The network N_1 has the four vertices

$$
\begin{aligned}
p_0^1 &:= \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\Big)\,,\\
p_1^1 &:= \Big(\frac{1}{4}(\sqrt{3}-2), \frac{\sqrt{3}}{8}(\sqrt{2}-2), \frac{\sqrt{3}}{8}(\sqrt{2}-2)\Big)\,,\\
p_3^1 &:= \Big(\frac{1}{2}(\sqrt{3}-1), 0, \frac{\sqrt{3}}{2}\Big)\,,\\
p_2^1 &:= \Big(\frac{1}{4}(\sqrt{3}-2), -\frac{\sqrt{3}}{8}(\sqrt{2}-2), -\frac{\sqrt{3}}{8}(\sqrt{2}-2)\Big)\,,
\end{aligned}
$$

as well as the copies under the lattice

$$
p_4^1 = p_0^1 + g_1\,, \qquad p_5^1 = p_3^1 + g_2\,, \qquad p_6^1 = p_0^1 + g_3\,.
$$

We connect them with the six straight segments of Figure 3.10. The Λ_0-orbit then defines the network N_1, with quotient K_4. It can be checked by calculation that the balancing equation (3.2) holds at each of the vertices $p_0^1, \ldots, p_3^1$, and so N_1 is a Steiner network. The length of N_1 is

$$
\begin{aligned}
L(N_1) &= |p_1^1 - p_0^1| + |p_2^1 - p_1^1| + |p_2^1 - p_4^1| + |p_1^1 - p_5^1| + |p_2^1 - p_3^1| + |p_6^1 - p_3^1|\\
&= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}(\sqrt{2}-1) + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}(\sqrt{3}-1)\\
&= \frac{\sqrt{3}}{2}(2+\sqrt{2}+\sqrt{3}).
\end{aligned}
$$

We define N_t for $t \in (0,1)$ as a convex combination of the vertices of N_0 and N_1,

$$
p_i^t := (1-t)p_i^0 + tp_i^1\,, \qquad i = 1, \ldots, 6\,,
$$

again connected with six straight segments as in Figure 3.10.

The fact that $L(N_t)$ is a decreasing function of $t \in [0,1]$ follows from three facts. First, $L(N_1)$ is strictly less than $L(N_0) = 9/2$. Second, the function $t \mapsto L(N_t)$ is critical at $t = 1$ since N_1 is Steiner. Third, each term $t \mapsto |p_i^t - p_j^t|$ of $L(N_t)$ is convex on $[0,1]$, and so is $t \mapsto L(N_t)$. □

4 Periodic networks of fixed degree minimizing length

In this chapter we minimize the length quotient of periodic networks with prescribed topology. We call an n-periodic network *of degree d* if all vertices have degree d, that is, the base graph G_0 is a d-regular graph.

Clearly, length criticality is equivalent to force balancing. In particular, all vertices of a minimizer for L^n/V are balanced. We want to analyse networks with the simplest topology:

Definition 4.1. We call an n-periodic network N of degree d *irreducible* if its quotient N/Λ has the least number of vertices possible for a balanced network of degree d in $\mathbb{R}^n$.

Our goal is to classify the topology of irreducible networks. Irreducibility can be related to the circuit rank of the base graph G_0. An n-periodic network N of degree d must have $2\#\text{edges} = d\#\text{vertices}$ and so $\operatorname{rank} N = 1 + \left(\frac{d}{2} - 1\right)\#\text{vertices}$. Therefore an n-periodic network of degree d satisfies

$$\#\text{vertices} \geq \frac{2n-2}{d-2}; \tag{4.1}$$

in particular, a network with degree $d < 2n$ has at least two vertices.

Remark 4.2. For the Steiner case, $d = 3$, an n-periodic network N is irreducible if and only if $\operatorname{rank} N = n$. Indeed, a balanced network with one vertex and n loops in the quotient can be split into a balanced Steiner network on $2n - 2$ vertices, see Theorem 3.9. For $d > 3$, however, $\operatorname{rank} N$

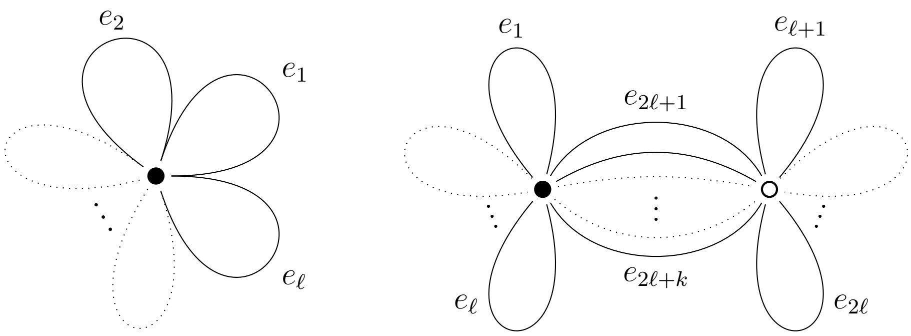


Figure 4.1: The bouquet graph B_ℓ (left) and the double bouquet graph $D_{\ell,k}$ (right). The latter is obtained by joining two copies of B_ℓ with k edges. As stated in Proposition 4.4, irreducible networks of degree $d > n$ cover one of these graphs.

can be larger than n: Proposition 4.4 below gives an irreducible example with $d = 5$, $n = 3$ on 2 vertices, so that the rank is 4.

4.1 Topology of irreducible n-periodic networks

One or two vertices are clearly the simplest case for the topology of a quotient graph. Our goal is to show that no more vertices are needed for networks of degree $d > n$ to be irreducible. We start by introducing connected multigraphs with one or two vertices, see Figure 4.1:

- The *bouquet graph* of order ℓ, denoted by B_ℓ, consists of one vertex with $\ell \geq 0$ loops and has degree 2ℓ.
- The *double bouquet graph* of order (ℓ, k), denoted by $D_{\ell,k}$, consists of two bouquet graphs of order $\ell \geq 0$, connected with $k \geq 1$ edges. It has degree $2\ell + k$. Specifically, we call $D_k := D_{0,k}$ the *dipole graph* of order k. That is, $D_{\ell,k} := D_k \,\Box\, B_\ell$.

We begin with an existence statement.

Lemma 4.3. *Let $n \geq 2$ and $\Lambda \subset \mathbb{R}^n$ be a lattice. There exist balanced n-periodic networks of degree d;*

(i) for $d \geq n+1$ such that the quotient is a double bouquet graph, and

(ii) specifically for even $d \geq 2n$ such that the quotient is $B_{d/2}$.

Proof. We distinguish three cases to construct the graphs; compare with Figure 4.2.

(ii): Suppose d is even and $d \geq 2n$. To define N pick first a point $p \in \mathbb{R}^n$ and connect it with points of $p + (\Lambda \setminus \{0\})$ with edges as follows. Choose n vectors generating the lattice Λ, and use them to define a set of $n \leq d/2$ edges. Supplement this edge set in a way that the resulting set of $d/2$ edges does not contain any pair of parallel edges, in particular no opposite ones. Then take the Λ-orbit of this edge set to define a network N of degree d, which is balanced and has rank n; moreover, the star of p is embedded, implying that N is immersed. Observe the quotient graph N/Λ is topologically $B_{d/2}$.
(i), Case 1: Suppose $d \geq n+1$ is odd. We construct a network of degree d with topology $D_{(d-3)/2,3}$. The network turns out to be a generalization of the bnn network, see Figure 4.8. Suppose Λ is generated by $g_1, \ldots, g_n$. Let P be the plane in $\mathbb{R}^n$ spanned by g_1 and g_2.
In a first step we construct a balanced network of degree 3 in P with quotient D_3. We may assume the two generators g_1, g_2 of $\Lambda_0 := \Lambda \cap P$ are chosen to enclose an angle in $[\pi/3, \pi/2]$. Then the triangle with vertices

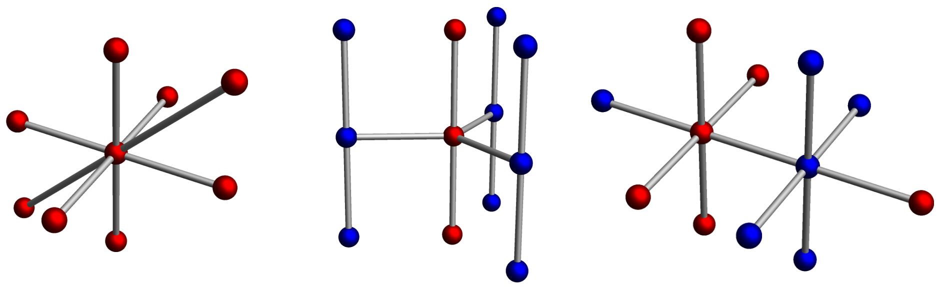


Figure 4.2: Construction of balanced n-periodic networks with prescribed degree. The figures correspond to the three cases in the proof of Lemma 4.3: shown are $d = 8$, $d = 5$, $d = 6$ for $n = 3$, with quotient graphs B_4, $D_{1,3}$, $D_{2,2}$, respectively.

$0, g_1, g_2$ contains a Fermat point q in its interior, and so the three edges connecting q to 0, g_1, g_2 are balanced at q.
Let N_2 be the Λ_0-orbit of this tripod, which is an embedded network of degree 3 with topology D_3. Note that N_2 is balanced, since a network with two vertices in the quotient is balanced on both vertices if it is balanced at one vertex.
The second step is a product construction similar to our proof of (ii). Let Λ_1 be the sublattice generated by $g_3, \ldots, g_n$. Connect the vertex $0 \in \Lambda$ with $(d-3)/2$ vertices in $\Lambda \setminus \Lambda_0$ and also the vertex q with $(d-3)/2$ vertices in $q + (\Lambda \setminus \Lambda_0)$, again subject to the following condition: the altogether $d-3$ edges extend the set of $n-2$ vectors $g_3, \ldots, g_n$ in a way that neither at 0 nor at q there is a pair of parallel edges. Note that by assumption $d-3 \geq n-2$. Finally, we let N be the Λ-orbit of these edges, so that all edges incident to a point either have direction in P or occur in opposite pairs. Then N is balanced and immersed, and the lattice Λ of N has rank n and degree d.
(i), Case 2: Assume $d \geq n+1$ is even. We construct a network N of degree d covering $D_{(d/2)-1,2}$. Pick a generator g_1 of Λ and consider the edge e from 0 to g_1. Then choose q in the interior of e and proceed as in the second step of Case 1: Connect each of $0, q$ to a point in $\Lambda \setminus \mathbb{Z}g_1$ by $\frac{d}{2}-1$ edges, this time including the directions of the remaining $n-1$ lattice vectors into the total edge set (thereby using the assumption $d-2 \geq n-1$). Again the Λ-orbit N of this edge set satisfies all requirements. Note that this construction would coincide with the one for (ii) if q were 0. $\square$

If an n-periodic network in $\mathbb{R}^n$ has degree 3 it must have a quotient with at least $2n-2$ vertices. Thus the topology becomes more complex with increasing dimension n. In contrast, for sufficiently high degree d the lemma implies that irreducible networks have a simple topology:

Proposition 4.4. *Let N be an irreducible n-periodic network of degree $d \geq n+1$. If d is even and $d \geq 2n$, then N covers $B_{d/2}$. For all other $d \geq n+1$, the network N covers one of the graphs $D_{\ell,k}$ where $2 \leq k \leq d$*

and $d - k = 2\ell$.

Proof. For even $d \geq 2n$, Lemma 4.3 (ii) asserts the existence of a network whose quotient $B_{d/2}$ has one vertex, and so is irreducible. A finite graph with one vertex necessarily has even degree. For odd $d \geq n + 1$, networks with two vertices exist by part (i), so that networks with two vertices are indeed irreducible.

A rank consideration shows that for even d with $n+1 \leq d < 2n$ networks with two vertices are irreducible, too: On the contrary, suppose the quotient has only one vertex, i.e., it is $B_{d/2}$. Since the quotient of an n-periodic network has rank at least n, this gives $n \geq \operatorname{rank} B_{d/2} = d/2$, ruling out this case.

Finally a quotient $D_{\ell,k}$ with $k = 1$ is impossible, as an immersion covering $D_{\ell,1}$ cannot be balanced. □

Remark 4.5. For d odd the number of graphs which are admissible for the Proposition is $\lfloor d/2 \rfloor$ and so increases with d. We should expect that minimizers favour a small number of loops ℓ, since it seems easier to make the k "bridges" short. Nevertheless we will see that for $n = 3$ and $d = 5$ the quotient graph $D_{1,3}$ leads to a shorter minimizer than D_5.

4.2 Networks of degree $d = n + 1$

We want to determine optimal n-periodic networks of degree $n + 1$. For dimension $n = 3$ this is the simplest case which is not Steiner. The minimizer among irreducible triply networks of degree $d = 4$ will turn out to be the well-known diamond network, which can be characterized by the fact that the neighbours of each vertex form the vertices of a regular tetrahedron.

In the present section we obtain the same characterization in arbitrary dimension: The minimizers among irreducible n-periodic networks of degree $d = n + 1$ are networks N for which each vertex $q \in N$ is the center of symmetry of a regular n-simplex, defined by the neighbours of q. This will be shown in Theorem 4.8.

Our first goal is an estimate on the length for a graph G_0 connecting the origin to the vertices of an arbitrary simplex Δ:

Proposition 4.6. *Let Δ be an n-simplex with vertices $p_0, \dots, p_n \in \mathbb{R}^n$ and volume $V(\Delta) > 0$. Then*

$$\frac{(L(G_0))^n}{V(\Delta)} \geq n! \sqrt{(n+1)^{n-1}\, n^n}\,, \tag{4.2}$$

where we set $L(G_0) := \sum_{i=0}^n |p_i|$. Equality holds if and only if Δ is a regular n-simplex with symmetry centre the origin.

The proof will depend on an estimate, which we formulate for a pyramid generalizing the simplex so that we can make use of it also in the proof of Theorem 4.16 below.

Consider a convex polyhedron E contained in the hyperplane $P := \mathbb{R}^{n-1} \times \{0\}$, such that E has $k \geq n \geq 2$ pairwise distinct vertices $p_1, \dots, p_k \in P$. We assume E has positive $(n-1)$-dimensional volume $V_E > 0$. We then take a pyramid $\Delta \subset \mathbb{R}^n$ with base E and apex $p_0 \in \mathbb{R}^n \setminus P$ as in Figure 4.3. We denote with $V(\Delta) > 0$ its n-dimensional volume. Moreover, we consider an arbitrary point $q \in \mathbb{R}^n$ and a graph G_q which is the union of the edges from q to the vertices $p_0, \dots, p_k$ of Δ. We denote its length by $L(G_q)$, and the total length of the edges from q to the base vertices by $s := \sum_{i=1}^k |p_i - q| > 0$.

Lemma 4.7. *For given $p_0, p_1, \dots, p_k$ and each $q \in \mathbb{R}^n$ the length of G_q satisfies*

$$\frac{(L(G_q))^n}{V(\Delta)} \geq \frac{n^2}{V_E} \left(\frac{k^2-1}{k^2} \frac{ns}{n-1} \right)^{n-1}. \tag{4.3}$$

The equality case is equivalent to the following conditions: $p_0 - q$ is perpendicular to P, as well as

$$|p_1 - q| = \dots = |p_k - q| = \frac{k(n-1)}{k^2-n} |p_0 - q|\,, \quad \textit{and} \quad \operatorname{dist}(q, P) = \frac{s}{k^2}\,. \tag{4.4}$$

Proof. Set $x_i := |p_i - q|$ for $i = 1, \dots, k$, and $z := |p_0 - q|$. Then

$$L(G_q) = \sum_{i=1}^{k} x_i + z = s + z\,,$$

which is positive due to $s > 0$. Setting $h := \mathrm{dist}(q, P) \geq 0$ we can estimate the volume of the pyramid Δ by

$$V(\Delta) \leq \frac{1}{n}(h + z)V_E\,, \tag{4.5}$$

where equality is attained if and only if $p_0 - q$ is perpendicular to P and q lies in the closed slab of $\mathbb{R}^n$ between P and p_0. Note that $p_0 \notin P$ implies $h + z > 0$. Therefore, an equivalent inequality is

$$\frac{(L(G_q))^n}{V(\Delta)} \geq \frac{n(s+z)^n}{V_E(h+z)}\,. \tag{4.6}$$

For a moment, let us regard the right-hand side of (4.6) as a function of $z \in (-h, \infty)$; due to $h < s$ this function is positive. Differentiation yields the unique critical point

$$z_0 := \frac{s - nh}{n - 1}\,.$$

As z tends to $-h$ or to infinity, the right hand side of (4.6) tends to infinity, and so z_0 assigns a minimum to the right hand side of (4.6). But $s > kh \geq nh$ implies $z_0 > 0$ so that we have shown that for $z \in (0, \infty)$ the right-hand side of (4.6) takes a unique strict minimum at z_0.

Inserting z_0 into the inequality (4.6) yields

$$\frac{(L(G_q))^n}{V(\Delta)} \geq \frac{n^2}{V_E}\left(\frac{n(s-h)}{n-1}\right)^{n-1}\,. \tag{4.7}$$

In particular, (4.3) holds strictly in case $h = 0$, implying the lemma for this case. Thus we may assume $h > 0$ in the following.

The existence of z_0 means that there exists a $q \in \mathbb{R}^n$ minimizing the quotient $(L(G_q))^n/V(\Delta)$ for the given p_i's. The equality discussion for (4.6)

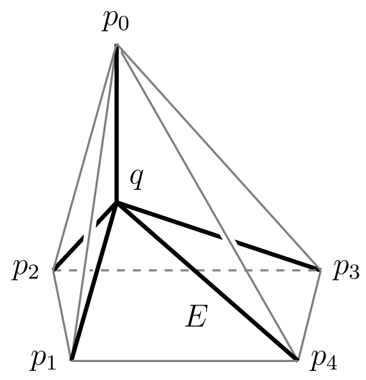


Figure 4.3: The pyramid Δ of Lemma 4.7 with base E and apex p_0. The graph G_q connects the vertices of Δ with a further point q.

implies we must have $p_0 - q \in P^\perp$, and since (4.6) has a strict mimimum at $z_0 > 0$, we have $|p_0 - q| = z_0 > 0$, and so $q \neq p_0$, in particular.

The volume $V(\Delta)$ is independent of q, so that q also minimizes $L(G_q)$. Since all edge lengths are positive, G_q must be balanced at q. The balancing formula (3.2) gives $\sum_{i=1}^k h/x_i = 1$. This harmonic mean can be estimated by an arithmetic mean,

$$h = \Big(\sum_{i=1}^k \frac{1}{x_i} \Big)^{-1} \leq \frac{s}{k^2}\,, \tag{4.8}$$

where equality holds if and only if $x_1 = \cdots = x_k$. Combining (4.7) and (4.8) yields the desired estimate (4.3).

Finally, the equality statement (4.4) follows from considering the equality cases in (4.5), (4.7) and (4.8): $(L(G_q))^n/V(\Delta)$ is minimal if and only if $p_0 - q \in P^\perp$, $z = z_0$, and $x_1 = \ldots = x_k$, so that for all $i = 1, \ldots, k$

$$h = \frac{s}{k^2} = \frac{1}{k}|p_i - q| \quad \text{and} \quad z = \frac{s - nh}{n-1} = \frac{k^2 - n}{k(n-1)}|p_i - q|\,. \qquad \square$$

Proof of Proposition 4.6. The left-hand side of (4.2) is scaling invariant so we may assume $V(\Delta) = 1$. Moreover, $L(G_0)$ is a continuous function of $p_0, \ldots, p_n \in \mathbb{R}^n$, and a minimizing sequence for $L(G_0)$ clearly has all $|p_i|$

bounded. Thus a minimizer Δ for $(L(G_0))^n/V(\Delta)$ exists.

We want to show that Δ is regular. For arbitrary $0 \leq \ell \leq n$, regard the simplex Δ as a pyramid with apex p_ℓ and apply Lemma 4.7 with $k = n$ and $q = 0$. The first equations of (4.4) give

$$|p_0| = \cdots = |p_n|\,,$$

while the perpendicularity of p_ℓ to the hyperplane containing the other vertices gives

$$0 = \langle p_\ell, p_i - p_j\rangle = \langle p_\ell, p_i\rangle - \langle p_\ell, p_j\rangle \quad \text{for all } i, j \neq \ell\,.$$

We conclude the $n+1$ vertices are contained in a sphere and make pairwise equal angles when viewed from the origin. Hence Δ is a regular simplex as stated.

For a regular n-simplex Δ, length and volume can be computed as the following functions of the edge length a,

$$L(G_0) = a\,\frac{(n+1)}{\sqrt{2}}\sqrt{\frac{n}{n+1}} \quad \text{and} \quad V(\Delta) = \frac{a^n}{n!}\sqrt{\frac{n+1}{2^n}}\,.$$

Inserting these values into (4.2) gives the desired estimate. □

From the proposition we now derive an existence and uniqueness statement which in particular applies to degree-4 networks in $\mathbb{R}^3$ or doubly periodic Steiner networks in $\mathbb{R}^2$.

Theorem 4.8. *Let N be an irreducible n-periodic network of degree $d = n+1$ for $n \geq 2$. Then its length quotient satisfies*

$$\frac{L^n}{V} \geq \sqrt{(n+1)^{n-1}n^n}\,. \tag{4.9}$$

Equality holds if and only if N covers the dipole graph D_{n+1} and for each vertex $q \in N$ the leaves of star q *form the vertices of a regular n-simplex.*

For $n = 3$ this proves the standard diamond network with degree $d = 4$ minimizes the length quotient, with $L^3/V = \sqrt{4^2\,3^3} = 12\sqrt{3}$. See Section 4.4 for a complete discussion of the case $d = 4$ in $\mathbb{R}^3$. Let us also note that for $n = 2$ the Theorem confirms the optimality of the hexagonal hcb network, a fact we proved in Proposition 3.15.

Proof. By Proposition 4.4, the network N covers the double bouquet graph $D_{\ell,k}$ for some $2 \le k \le n+1$ with $2\ell + k = n+1$. Note that $D_{\ell,k}$ contains exactly

$$2\ell + (k-1) = n = \operatorname{rank} N \tag{4.10}$$

cycles generating the first homology group; since N is n-periodic they are independent, that is, each lifts to an independent generator of Λ.

We remove from N all edges projecting to the loops of $D_{\ell,k}$. From the remaining subset we consider a component $N' \subset N$. The graph N' covers D_k and so has degree k. Moreover, since each cycle of $D_{\ell,k}$ is independent, so is each of the $k-1$ generating cycles of D_k. Consequently N' is a $(k-1)$-periodic network, contained in some $(k-1)$-dimensional affine subspace of $\mathbb{R}^n$.

Let L' denote the length of N' and V' be its $(k-1)$-dimensional volume. We claim

$$\frac{L^n}{V} \ge \frac{n^n}{(k-1)^{k-1}} \frac{(L')^{k-1}}{V'} \qquad \text{for} \quad k = 2, \dots, n+1. \tag{4.11}$$

In case $k = n+1$ the quotient N/Λ has no loops, so that $N' = N$ and (4.11) is immediate. Thus consider the case $k \le n$. Each of the 2ℓ loops of $D_{\ell,k}$ gives rise to a generator of Λ, not contained in Λ'. Moreover, the loops lift to straight edges $e_1, \dots, e_{2\ell}$ of N which are not contained in N'. These edges contribute length to N, but not to N', and we can estimate

$$\frac{L^n}{V} \ge \frac{\Big(L' + |e_1| + \cdots + |e_{2\ell}|\Big)^n}{V'\,|e_1| \cdots |e_{2\ell}|}\,. \tag{4.12}$$

In terms of $x := \sqrt[2\ell]{|e_1| \cdots |e_{2\ell}|} > 0$ the estimate on geometric and arithmetic mean yields

$$\frac{L^n}{V} \geq \frac{\big(L' + 2\ell x\big)^n}{V' x^{2\ell}} = \frac{\big(L' + (n+1-k)x\big)^n}{V'\, x^{n+1-k}}\,. \tag{4.13}$$

Regard the right-hand side of (4.13) as a function of $x \in (0, \infty)$, and differentiate to find the unique critical point at $x_0 = L'/(k-1)$. Moreover, the limit $x \to 0$ verifies that x_0 is a minimum. Insertion of x_0 into (4.13) proves our claim (4.11).

We want to derive an explicit estimate from (4.11) which will show that L^n/V can be estimated by its minimal value for $k = n + 1$. Pick a vertex $q \in N'$. Its k neighbours $p_1, \ldots, p_k \in N'$ form the vertices of a $(k-1)$-simplex Δ (that is, a pyramid) with volume

$$V(\Delta) = \frac{V'}{(k-1)!}\,.$$

The length L' of N' coincides with the length of $\operatorname{star} q$. We apply Proposition 4.6 to Δ and conclude that the length quotient L'^{k-1}/V' is minimal if and only if Δ is a regular $(k-1)$-simplex with q the center of symmetry. Estimating the right hand side of (4.11) with (4.2) (for $n = k - 1$) gives

$$\frac{L^n}{V} \geq n^n \sqrt{k^{k-2}(k-1)^{1-k}} \qquad \text{for} \quad k = 2, \ldots, n+1. \tag{4.14}$$

The right-hand side of (4.14) is strictly decreasing in k, and so L^n/V can be estimated by the right hand side with $k = n + 1$; in particular, (4.9) holds.

Equality in (4.14) (and so in (4.9)) can only hold for $k = n + 1$, in which case $N' = N$ and N covers D_{n+1}. Our derivation shows that for $k = n + 1$ equality in (4.9) holds precisely for the case that Proposition 4.6 holds with equality, namely for a regular n-simplex with q the centre of symmetry. $\square$

Remark 4.9. For $2n > d > n + 1$ an optimal n-periodic network of degree d does not necessarily cover D_d. For example, in dimension $n = 3$ the

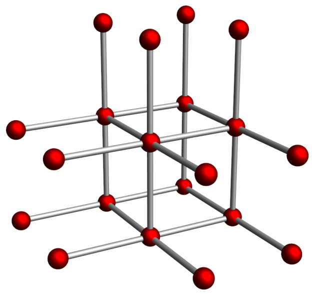


Figure 4.4: Among triply periodic networks of degree 6, the pcu network with quotient B_3 minimizes the length quotient.

minimizer for L^3/V among networks of degree $d = 5$ is the bnn network, covering $D_{1,3}$ (cf. Table 1).

4.3 Networks of degree $d \geq 2n$

By Proposition 4.4, an irreducible network of even degree $d \geq 2n$ covers the bouquet graph $B_{d/2}$. We estimate its length quotient:

Theorem 4.10. *Let N be an irreducible n-periodic network of even degree $d \geq 2n$ with lattice Λ. Then*

$$\frac{L^n}{V} \geq \left(\frac{d}{2} - n + 1\right) n^n . \tag{4.15}$$

Equality holds if and only if $d = 2n$ and Λ is similar to the primitive lattice $\mathbb{Z}^n$.

For $\mathbb{R}^3$ this settles the case $d = 6$: equality is attained by the pcu network which has the edge set of a tessellation of 3-space with cubes (see Figure 4.4). Similarly, for $n = 2$, the sql network relating to a square tessellation is optimal. The estimate (4.15) implies that in each dimension n networks with even degree $d > 2n$ have a length quotient larger than for $d = 2n$.

Proof. Pick a vertex $p_0 \in N$. We consider its neighbours $q_1, \ldots, q_d$ and set $g_i := q_i - p_0$. Since N covers the graph $B_{d/2}$ we may assume the indexing is such that the $n \leq d/2$ vectors $g_1, \ldots, g_n$ span Λ, that $L = \sum_{i=1}^{d/2} |g_i|$, and that $|g_1| = \min_{1 \leq i \leq d/2} |g_i|$. The inequality on geometric and arithmetic mean gives

$$\begin{aligned} V = |\det(g_1, \ldots, g_n)| &\leq \frac{1}{\frac{d}{2} - n + 1} \left(\frac{d}{2} - n + 1\right)|g_1| \cdot |g_2| \cdots |g_n| \\ &\leq \frac{1}{\left(\frac{d}{2} - n + 1\right)n^n} \left(\left(\frac{d}{2} - n + 1\right)|g_1| + |g_2| + \ldots + |g_n|\right)^n . \end{aligned} \tag{4.16}$$

Moreover, we use $n \leq d/2$ and $|g_1| \leq |g_i|$ for $i = n+1, \ldots, d/2$ to obtain

$$V \left(\frac{d}{2} - n + 1\right) n^n \leq \left(|g_1| + |g_2| + \ldots + |g_{d/2}|\right)^n = L^n . \tag{4.17}$$

Let us show that equality cannot hold for $d \geq 2n + 2$. If the second inequality of (4.16) happens to be an equality, then

$$\left(\frac{d}{2} - n + 1\right)|g_1| = |g_2| = \ldots = |g_n| .$$

In particular, $|g_2|, \ldots, |g_{d/2}|$ are strictly larger than $|g_1|$ and equality cannot hold in (4.17). For $d = 2n$, however, equality holds if and only if $g_1, \ldots, g_n$ are pairwise perpendicular and have the same length, i.e., Λ is the primitive n-dimensional lattice. $\square$

Remark 4.11. The construction of the proof of Theorem 4.10 shows the length quotient of irreducible n-periodic networks is strictly increasing when restricted to even degree $d \geq 2n$: Removal of an edge of the quotient network N/Λ and thereby degree reduction by 2 decreases length while not affecting balancing.

The theorem leaves open the case of networks with odd degree. We present an estimate for the length quotient for that case, which is weaker than (4.15):

Theorem 4.12. *If N be an irreducible n-periodic network of degree $d \geq n+1$*

then

$$\frac{L^n}{V} > \sqrt{(n+1)^{n-1}\, n^n}\,. \tag{4.18}$$

Equality holds if and only if N covers the dipole graph D_{n+1} and for each vertex $q \in N$ the leaves of $\operatorname{star} q$ *form the vertices of a regular n-simplex.*

For $n = 3$ we will obtain a stronger estimate in Corollary 4.17.

Proof. For $n \geq 2$ and even degree $d \geq 2n$, Theorem 4.10 gives

$$\frac{L^n}{V} \geq \Big(\frac{d}{2} - n + 1\Big) n^n \geq n^n\,.$$

On the other hand,

$$\sqrt{(n+1)^{n-1}\, n^n} = \sqrt{\frac{(n+1)^n}{n+1}\, n^n} \leq n^n \sqrt{\frac{1}{3}\Big(1 + \frac{1}{n}\Big)^n}\,,$$

and so $(1 + 1/n)^n \leq 3$ implies (4.18).

For all other $d \geq n + 1$, Proposition (4.4) identifies the topology of N/Λ as a double bouquet graph $D_{\ell,k}$. If k is odd we have $k \geq 3$. Pick an n-periodic subnetwork $N' \subset N$ (with the same lattice) subject to the following property: The removal of any edge from N'/Λ disconnects the covering network N'. Then N'/Λ decomposes into two bouquet graphs of orders $\ell_1, \ell_2 \geq 0$, and $1 \leq k' \leq k$ edges connecting them. It contains exactly $\ell_1 + \ell_2 + (k' - 1) = n$ cycles. Clearly, $L(N) \geq L(N')$. If $d > n + 1$, the subnetwork N' is obtained by removing at least one edge from N. Thus $L(N) > L(N')$ for $d > n + 1$.

If $k' = 1$ then N' contains $\ell_1 + \ell_2 = n$ loops. Thus we can estimate the length of N'/Λ by a network covering the bouquet graph of order n. It has degree $2n$. Applying Theorem (4.10) yields (4.18) for this case. For $k' \geq 2$ we can follow the reasoning of the proof of Theorem 4.8, replacing 2ℓ by $\ell_1 + \ell_2$ and taking k' for k. This yields estimate 4.18 for N' and characterizes the equality case. $\square$

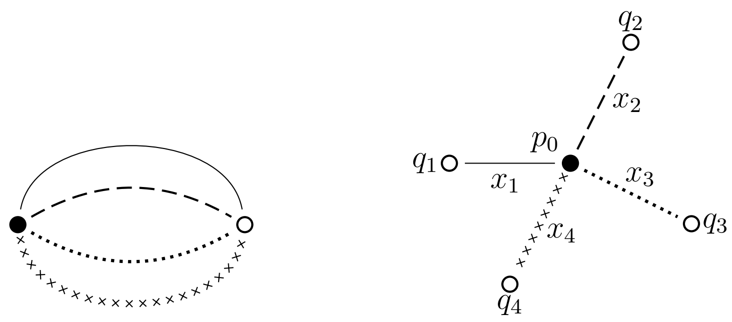


Figure 4.5: Topology and embedding of the dipole graph D_4.

4.4 Triply periodic networks of degree 4

In the remainder of this chapter, we study specifically the case of three dimensions. By Proposition 4.4, an irreducible triply periodic network of degree $d = 4$ must have a quotient with two vertices and four edges which is either the dipole graph D_4 or the double bouquet graph $D_{1,2}$. Theorem 4.8 asserts the absolute minimizer for the length quotient L^3/V with degree 4 covers D_4 and is the *diamond* network dia; it is uniquely determined up to similiarities of $\mathbb{R}^3$. This is included as part (i) of the following statement, while part (ii) determines the optimal embedding covering $D_{1,2}$, see Figure 4.6.

Theorem 4.13. *Let $N \subset \mathbb{R}^3$ be an irreducible triply periodic network of degree* 4.

(i) If N covers D_4, then

$$\frac{L^3}{V} \geq 12\sqrt{3} \approx 20.8\,. \tag{4.19}$$

Equality holds if and only if all edge lengths of N are equal and the lattice Λ is face-centered cubic, i.e., precisely for the diamond network dia.

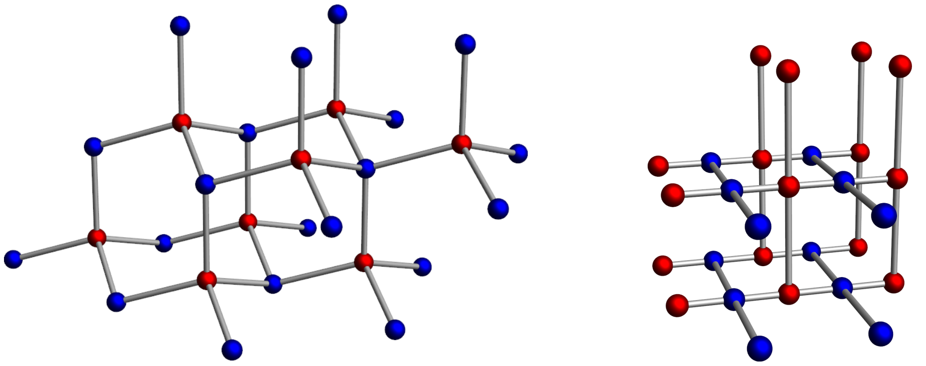


Figure 4.6: Among irreducible triply periodic networks of degree 4, the diamond network shown left minimizes the length quotient; it covers the dipole graph D_4. The other graph with degree 4 on two vertices is $D_{1,2}$; a minimizing cds network is shown on the right.

(ii) If N covers $D_{1,2}$, then

$$\frac{L^3}{V} \geq 27\,. \tag{4.20}$$

Up to similiarity, equality is attained by a 1-parameter family of networks with primitive lattice Λ; we label these networks cds*.*

Proof. It remains to prove (ii). We take a subgraph of N consisting of two adjacent vertices p_1, q_1 and their neighbours p_2, p_3, q_2 so that the vertices

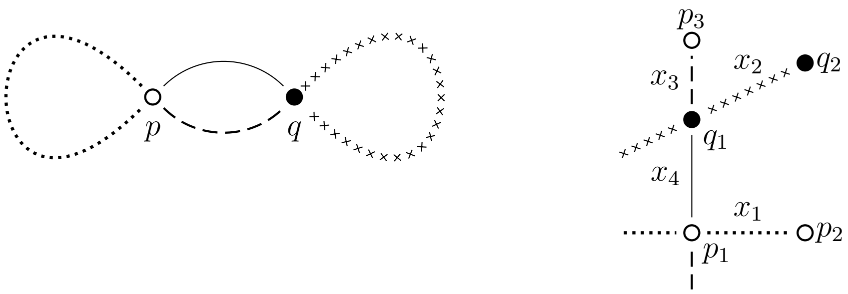


Figure 4.7: Topology and embedding of the double bouquet graph $D_{1,2}$.

p_1, p_2, p_3 and the vertices q_1, q_2 are identified in the lattice Λ, see Figure 4.7. The lattice Λ is generated by the lift of three loops of $D_{1,2}$, and so can be generated by

$$g_1 := p_2 - p_1\,, \qquad g_2 := q_2 - q_1\,, \qquad g_3 := p_3 - p_1.$$

The enclosed volume of N can be estimated by

$$\begin{aligned} V(\mathbb{R}^3/\Lambda) &= |\det(g_1, g_2, g_3)| \\ &\leq |g_1| \cdot |g_2| \cdot |g_3| \\ &\leq \frac{1}{27}\Big(|g_1| + |g_2| + |g_3|\Big)^3 \\ &\leq \frac{1}{27}\Big(|p_2 - p_1| + |q_2 - q_1| + |p_3 - q_1| + |q_1 - p_1|\Big)^3 \\ &= \frac{1}{27}L^3(N/\Lambda)\,. \end{aligned}$$

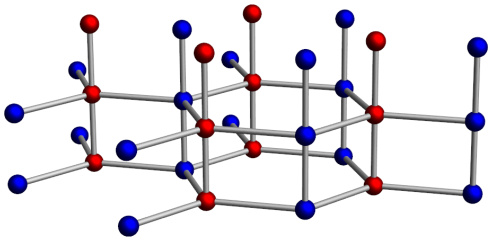
Equality holds if and only if the g_i's are pairwise perpendicular, have the same length and q_1 lies on the straight segment between p_1 and p_3. This implies the lattice is primitive and the edge lengths x_i given as in Figure 4.7 satisfy $x_1 = x_2 = x_3 + x_4 > 0$. In particular, equality for a fixed volume constraint $V = 1$ is attained by a 1-parameter family, parameterized by $x_4 \in (0, x_2)$, say. $\square$

A cds network with $x_3 = x_4$ is shown in Figure 4.6. In the two limits $x_3 \to 0$ and $x_4 \to 0$ the cds network degenerates to the pcu network of degree 6.

Remark 4.14. The network shown in Figure 4.6 is also the unique minimizer of the energy $\left(\sum x_i^2\right)^{3/2}/V$, see Sunada's book [Sun12].

4.5 Triply periodic networks of degree 5

Determining an optimal network of degree 5 is more difficult than the case of degree 4. This is due to the fact that an irreducible quotient graph Γ has



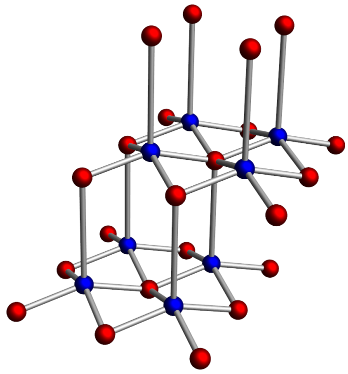


Figure 4.8: Among all irreducible triply periodic networks of degree 5, the bnn network with quotient $D_{1,3}$ minimizes the length quotient (left). The other possible graph of order 5 is the dipole D_5, for which the sqp network minimizes (right).

5 edges and so its fundamental group is generated by 4 elements. Thus, one of the generators for N must be contained in the lattice generated by the other three. This presents an integer constraint for our length optimization problem.

According to Proposition 4.4, an irreducible network of degree 5 can only attain the topologies D_5 or $D_{1,3}$. The network with smallest length quotient turns out to be a network covering $D_{1,3}$, which we call bnn. It corresponds to the edges of a tessellation of $\mathbb{R}^3$ with hexagonal prisms, i.e., it contains parallel layers of minimizing doubly periodic hexagonal networks, see Figure 4.8.

Theorem 4.15. *If N is an irreducible triply periodic network of degree 5 covering $D_{1,3}$ then*

$$\frac{L^3}{V} \geq 27\sqrt{3} \approx 46.8\,. \tag{4.21}$$

In the equality case, N is the bnn *network with a hexagonal lattice: the network consists of prismatic honeycombs over regular hexagons, where the prism height equals $3/4$ of the hexagon edge length.*

Proof. Consider two vertices, labelled $p_0, q_1 \in N$, which project to the two distinct vertices p, q of $D_{1,3}$. Consider first the neighbours of the point p_0, see Figure 4.9. The loop endpoints in $D_{1,3}$ correspond to two neighbours p_1, p_2 of p_0, which project again to p. The three edges of $D_{1,3}$ give rise to three further neighbours q_1, q_2, q_3, projecting onto q. The edges from p_0 to p_1 and p_2 are opposite at p_0 and contained in a line ℓ.

We claim that it is sufficient to verify the theorem for N balanced. Note first that for a network with two vertices in the quotient, balancing at one vertex is equivalent to balancing at the other vertex. Suppose now N is not balanced. Then N is not balanced at p_0, and so replacing p_0 with the Fermat point F of the triangle q_1, q_2, q_3 yields a balanced network with strictly smaller length, but with the same lattice and volume. Possibly, the resulting network is not immersed, namely in case q_1, q_2, q_3 are collinear, or the triangle q_1, q_2, q_3 has an interior angle of at least 120 degrees. In that case, however, F coincides with one of the vertices q_1, q_2, q_3, and so N can be regarded as a network covering the bouquet graph B_4. Applying Theorem (4.10) gives $L^3/V \geq (4-3+1)\, 3^3 = 54$, so that (4.21) holds strictly.

Balancing at p_0 implies that q_1, q_2, q_3 must be coplanar with p_0, thereby defining a plane P. The same reasoning applies to the three neighbours of q_1 projecting to p, they define a plane P'. The edge triples defining P and P' agree up to the translation from p_0 to q_1, and so $P = P'$.

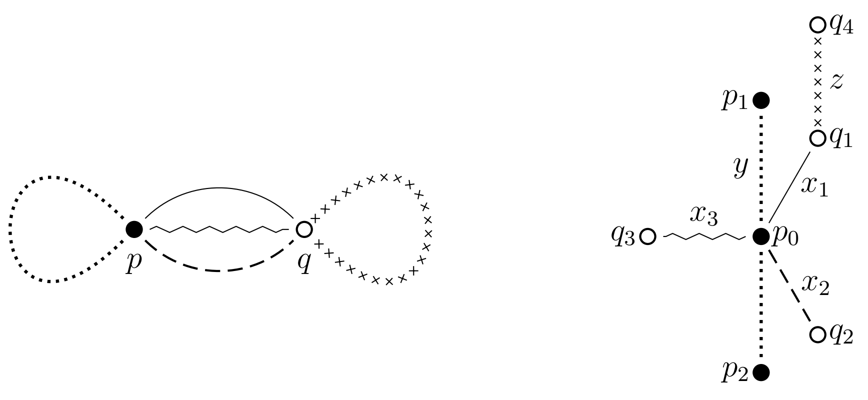


Figure 4.9: Topology and embedding of the double bouquet graph $D_{1,3}$.

Consider now the line ℓ' through q_1 determined by its two neighbours projecting to p. For the lattice to have rank 3, at least one of the lines ℓ, ℓ' must be transverse to the plane P. Hence $p_1 - p_0$ or $q_4 - q_1$ is a generator of the lattice. By relabelling let us assume $p_1 - p_0$ has this property.

The points q_1, q_2, q_3 are not collinear and define a triangle with positive area A_Δ. Thus the volume V of N/Λ satisfies

$$V \le 2A_\Delta \operatorname{dist}(p_1, P)\,, \tag{4.22}$$

Equality in (4.22) is attained if and only if from the four generators of the homology of $D_{1,3}$,

$$g_1 := q_1 - q_3\,, \quad g_2 := q_2 - q_3\,, \quad g_3 := p_1 - p_0\,, \quad g_4 := q_4 - q_1\,,$$

the first three span the lattice Λ.

Setting $x_i := |q_i - p_0|$ for $i = 1, 2, 3$, and $y := |p_1 - p_0|$, $z := |q_4 - q_1|$ we have $L = x_1 + x_2 + x_3 + y + z$. We may assume a choice of coordinates with $p_0 = 0$ and

$$q_1 = x_1 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad q_2 = \frac{x_2}{2} \begin{pmatrix} 1 \\ \sqrt{3} \\ 0 \end{pmatrix}, \quad q_3 = \frac{x_3}{2} \begin{pmatrix} 1 \\ -\sqrt{3} \\ 0 \end{pmatrix}, \tag{4.23}$$

which gives

$$2A_\Delta = \Big|\det(q_1 - q_3,\, q_2 - q_3)\Big| = \frac{\sqrt{3}}{2}\Big(x_1x_2 + x_1x_3 + x_2x_3\Big)\,. \tag{4.24}$$

We now distinguish the case $q_4 \in P$ from $q_4 \notin P$.

Case 1: Suppose $q_4 \in P$ (cf. Figure 4.10). In $\mathbb{R}^3/\Lambda$ the vertex q_4 and q_1, q_2, q_3 are identified, and so in $\mathbb{R}^3$ the smallest lattice vector contained in P gives a lower bound for $|q_4 - q_1|$. For our hexagonal lattice $\Lambda \cap P$ this gives

$$|q_4 - q_1| \ge \min\Big\{|q_2 - q_1|, |q_3 - q_1|, |q_3 - q_2|\Big\}\,.$$

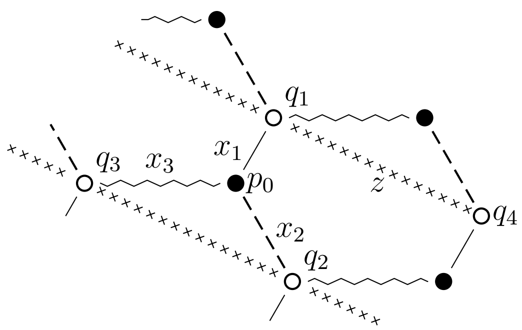


Figure 4.10: $P \cap N$ in Case 1, where q_4 is contained in the plane P spanned by q_1, q_2, q_3.

By relabelling we may assume $|q_4 - q_1| \geq |q_2 - q_1|$. This inequality and the geometric arithmetic mean inequality give

$$\begin{aligned} z \geq |q_2 - q_1| &= \sqrt{\Big(x_1 + \frac{1}{2}x_2\Big)^2 + \frac{3}{4}x_2^2} \\ &= \sqrt{\Big(x_1 + x_2\Big)^2 - x_1 x_2} \\ &\geq \frac{\sqrt{3}}{2}\Big(x_1 + x_2\Big). \end{aligned}$$

Thus we can estimate $s := x_1 + x_2 + x_3 + z$ as

$$s \geq \frac{2+\sqrt{3}}{2}(x_1 + x_2) + x_3\,.$$

Moreover, estimating $x_1 x_2$ in (4.24) gives

$$2A_\Delta \leq \frac{\sqrt{3}}{2}\left(\frac{1}{4}(x_1 + x_2)^2 + (x_1 + x_2)x_3\right).$$

We combine the last two inequalities to arrive at

$$\frac{s^2}{2A_\Delta} \geq \frac{\Big((2+\sqrt{3})(x_1 + x_2) + 2x_3\Big)^2}{2\sqrt{3}\Big(\frac{1}{4}(x_1 + x_2)^2 + (x_1 + x_2)x_3\Big)}\,. \tag{4.25}$$

Let us determine the minimum of the right-hand side of (4.25). Using scaling invariance of this quotient and $x_1 + x_2 > 0$ we may assume $x_1 + x_2 = 1$. So it suffices to minimize

$$x_3 \mapsto \frac{1}{2\sqrt{3}} \frac{\left(2+\sqrt{3}+2x_3\right)^2}{\frac{1}{4}+x_3} \qquad \text{for } x_3 > 0\,.$$

This function attains its minimal value $2(2+\sqrt{3})$ at $x_3 = (1+\sqrt{3})/2$, and so

$$\frac{s^2}{2A_\Delta} \geq 2\,(2+\sqrt{3})\,.$$

Inserting this estimate into (4.22) and then using an estimate on the geometric mean of the kind $a\left(\frac{b}{2}\right)^2 \leq \left(\frac{a+b}{3}\right)^3$ verifies (4.21) strictly (so that equality cannot be attained):

$$V \leq 2A_\Delta \operatorname{dist}(p_1, P) \leq \frac{1}{2(2+\sqrt{3})}(x_1+x_2+x_3+z)^2\, y \leq \frac{2}{27(2+\sqrt{3})} L^3\,.$$

Case 2: Suppose $q_4 \notin P$ so that q_4 lies in $\Lambda \setminus P$. Since g_1, g_2, g_3 generate the lattice the edge length z is at least $\operatorname{dist}(P, p_1)$, and so

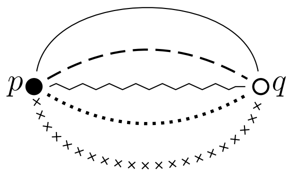
$$2\operatorname{dist}(p_1, P) \leq y + z\,. \tag{4.26}$$

On the other hand, estimating (4.24) by the arithmetic and geometric mean inequality gives

$$4\sqrt{3}\,A_\Delta = 3(x_1x_2 + x_1x_3 + x_2x_3) \leq (x_1+x_2+x_3)^2\,. \tag{4.27}$$

Then the inequality resulting from (4.22) and (4.24) can be estimated first using (4.26) and (4.27). Finally, the estimate on the geometric mean of the kind used before yields the desired inequality (4.21):

$$V \leq \frac{1}{4\sqrt{3}}(y+z)(x_1+x_2+x_3)^2 \leq \frac{1}{27\sqrt{3}} L^3\,. \tag{4.28}$$



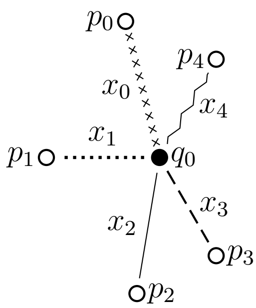


Figure 4.11: Topology and embedding of the dipole graph D_5.

Here equality can be attained: it holds if and only if

$$2y + 2z = 3x_1 = 3x_2 = 3x_3 \quad \text{and} \quad y = z = \operatorname{dist}(P, p_1) = \operatorname{dist}(P, q_4)\,,$$

so that N consists of parallel layers of honeycomb networks, connected orthogonally. □

We now discuss the other topology of irreducible networks of degree 5, namely the dipole graph D_5 as the quotient. Interestingly enough, like the double bouquet graph $D_{1,3}$, also D_5 can be covered by connected parallel layers of hexagonal networks. However, the distances between these layers cannot be chosen as in Theorem 4.15 because the four cycles generating the homology lead to a different integer constraint. The bnn network can be obtained as a network covering the dipole graph D_5. Its quotient graph, however, is always a covering graph of D_5 with more than two vertices. Hence another network arises as the optimal covering of D_5, called the sqp network:

Theorem 4.16. *Let N be a triply periodic network with quotient D_5. Then*

$$\frac{L^3}{V} \geq \frac{405}{8} = 50.625\,. \tag{4.29}$$

In case of equality the five neighbours of each vertex form the vertices of a square pyramid with height $L/3$.

Proof. Pick an arbitrary vertex $q_0 \in N$ together with its five neighbours $p_0, \ldots, p_4$. Note that L is the sum of the five edge lengths from q_0 to these points.

We consider first the case that there exists a plane P which contains four of the neighbours p_i. Then the fifth neighbour cannot be contained in P; we suppose it is labelled p_0. Moreover, we may assume the labelling is such that p_4 lies on the lattice spanned by p_1, p_2, p_3. Denote with T the triangle in P with vertices p_1, p_2, p_3.

The convex hull of the four points p_1 to p_4 is a triangle or a quadrilateral $E \subset P$. By our assumption and the fact that N is immersed, its area satisfies $\operatorname{area} E \geq 2 \operatorname{area} T$, where equality corresponds to E being a parallelogramme. Denote by Δ the pyramid with base E and apex p_0. The volume V of a fundamental domain for the lattice then is at most $3 \operatorname{vol} \Delta$. As in Lemma 4.7 we set $x_i := |p_i - q_0|$ for $i = 1, \ldots, 5$, and $s := x_1 + x_2 + x_3 + x_4$. The volume estimate and the lemma give

$$\frac{L^3}{V} \geq \frac{L^3}{3 \operatorname{vol} \Delta} \geq \frac{3}{\operatorname{area} E} \left(\frac{45}{32} \, s \right)^2. \tag{4.30}$$

Equality in (4.30) is equivalent to both inequalities attaining equality. The first inequality holds with equality if $\operatorname{area} E = 2 \operatorname{area} T$ so that E is a parallelogramme. Lemma 4.7 characterizes the case that the second inequality holds with equality: We must have

$$x_1 = x_2 = x_3 = x_4 = \frac{8}{13} x_0 \qquad \text{and} \qquad \operatorname{dist}(q_0, P) = \frac{1}{4} x_1, \tag{4.31}$$

as well as $p_0 - q_0$ perpendicular to P. Since (4.31) implies that p_1 to p_4 are contained in a circle in P, the parallelogramme must be a rectangle, and moreover p_0, q_0 project orthogonally onto its midpoint, having distances from P prescribed by (4.31).

Among the equality cases, (4.30) attains its minimum when the right hand side is minimal; moreover, this establishes a valid lower bound for the length quotient L^3/V. The only freedom is the conformal parameter of the

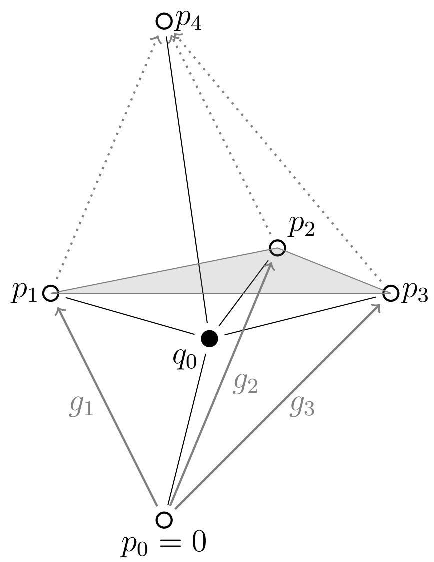


Figure 4.12: Notation for Theorem 4.16. Shown is a case where the vertices $p_0, \dots, p_4$ do not form a pyramid.

rectangle. Clearly, minimality of (4.30) occurs for maximal area E, i.e., for E a square. To compute (4.30) for this case note the diagonal of E has a length c satisfying

$$\left(\frac{c}{2}\right)^2 = x_1^2 - \left(\frac{x_1}{4}\right)^2 = \frac{15}{16}\,x_1^2, \qquad \text{and so} \qquad \text{area}\, E = 2\left(\frac{c}{2}\right)^2 = \frac{15}{8}\,x_1^2.$$

Inserting this expression into (4.30), thereby using $s = 4x_1$, gives the desired estimate (4.29) and verifies the claims for the equality case.

Now suppose no four p_i's are coplanar. We may assume that p_0 is the origin and the indexing is such that the lattice Λ is spanned by $g_i := p_i - p_0$ for $i = 1, 2, 3$, see Figure 4.12. Then p_4 is a lattice vector and so there are integer coefficients $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$ such that

$$p_4 = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 \,. \tag{4.32}$$

Let P_{123} be the plane through p_1, p_2, p_3 and consider the vector

$$n_{123} := (p_2 - p_1) \times (p_3 - p_1)$$

normal to P_{123}. The point p_4 has a signed distance from P_{123} given by $\mathrm{d}(p_4, P_{123}) = \langle n_{123}/|n_{123}|, p_4 - p_1\rangle$. Rewriting (4.32) as

$$p_4 - p_1 = (\lambda_1 + \lambda_2 + \lambda_3 - 1)p_1 + \lambda_2(p_2 - p_1) + \lambda_3(p_3 - p_1)\,,$$

we see the signed distance is

$$\mathrm{d}(p_4, P_{123}) = (\lambda_1 + \lambda_2 + \lambda_3 - 1)\mathcal{V}\,, \tag{4.33}$$

where $\mathcal{V} = \langle p_1, p_2 \times p_3\rangle = \det(p_1, p_2, p_3)$ is a signed volume of N/Λ. After relabeling we may assume that p_0 and p_4 lie on different sides of P_{123}, so that $\lambda_1+\lambda_2+\lambda_3 \geq 1$. Since no four p_i's are coplanar, in fact $\lambda_1+\lambda_2+\lambda_3 \geq 2$ and $\lambda_i \neq 0$ for $i = 1, 2, 3$. Moreover, we may assume p_1, p_2, p_3 are indexed such that $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

We now distinguish four cases for $(\lambda_1, \lambda_2, \lambda_3)$. In all cases there is a plane P through three of the p_i such that the remaining two vertices lie to opposite sides of P at different distances. In all cases, the result will be lower bound on L^3/V which is strictly larger than (4.29).

Case 1: Suppose $\lambda_1 + \lambda_2 + \lambda_3 \geq 3$. Equivalently, by (4.33), we have

$$2\,\mathrm{dist}(p_0, P_{123}) \leq \mathrm{dist}(p_4, P_{123})\,.$$

Thus, if A_{123} denotes the area of the triangle with vertices p_1, p_2, p_3 we find

$$V = 2A_{123}\,\mathrm{dist}(p_0, P_{123}) \leq 2A_{123}\frac{\mathrm{dist}(p_0, P_{123}) + \mathrm{dist}(p_4, P_{123})}{3}\,. \tag{4.34}$$

We set $x_i := |p_i - q_0|$ for $i = 0, \ldots, 4$, and claim

$$A_{123} \leq \frac{1}{\sqrt{3}}\left(\frac{x_1 + x_2 + x_3}{2}\right)^2. \tag{4.35}$$

To verify the claim, assume q_0 minimizes $x_1 + x_2 + x_3$. If q_0 coincides with p_3, then the estimate on geometric and arithmetic mean gives

$$A_{123} \leq \frac{1}{2} x_1 x_2 \leq \frac{1}{2} \Big(\frac{x_1 + x_2 + x_3}{2} \Big)^2 ,$$

thus proving (4.35). The same reasoning leads to (4.35) if $q_0 = p_1$ or $q_0 = p_2$. If, however, $q_0 \notin \{p_1, p_2, p_3\}$, the network is balanced at q_0. Then, choosing coordinates as in (4.23) leads to estimate (4.27). This proves the claim.
Inserting (4.35) into (4.34) gives

$$V \leq \frac{2}{3\sqrt{3}} \Big(\frac{x_1 + x_2 + x_3}{2} \Big)^2 (x_0 + x_4) \leq \frac{2}{81\sqrt{3}} L^3 .$$

This verifies (4.29).
Case 2: Suppose $\lambda_1 \leq -2$. We consider the plane P_{023} spanned by p_0, p_2, p_3 with normal vector $n_{023} := p_2 \times p_3$. Using (4.32), we have

$$|n_{023}| \, \mathrm{d}(p_1, P_{023}) = \langle n_{023}, p_1 \rangle = \mathcal{V} , \quad |n_{023}| \, \mathrm{d}(p_4, P_{023}) = \langle n_{023}, p_4 \rangle = \lambda_1 \mathcal{V} .$$

Since $\lambda_1 \leq -2$, the vertices p_1 and p_4 lie on opposite sides of P_{023} and

$$2 \operatorname{dist}(p_1, P_{023}) \leq \operatorname{dist}(p_4, P_{023}) .$$

As in Case 1 we obtain again $L^3/V \geq 81\sqrt{3}/2$.
Case 3: Suppose $\lambda_3 \geq 3$. Then, by (4.32)

$$|\det(p_1, p_2, p_4)| = \lambda_3 |\det(p_1, p_2, p_3)| \geq 3V .$$

Applying the estimate (4.19) on the length of a network covering D_4 to the subnetwork spanned by the four edges from q_0 to p_0, p_1, p_2, p_4 shows again that (4.29) holds strictly:

$$L^3 > (x_0 + x_1 + x_2 + x_4)^3 \geq 12\sqrt{3} \, |\det(p_1, p_2, p_4)| \geq 36\sqrt{3} \, V .$$

Case 4: Finally, assume $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$. In this case we consider the plane P_{014} spanned by p_0, p_1, p_4 with normal vector $n_{014} := p_1 \times p_4$. Then, by (4.32),

$$|n_{014}|\,\mathrm{d}(p_2, P_{014}) = \langle n_{014}, p_2\rangle = -2\mathcal{V}, \quad |n_{014}|\,\mathrm{d}(p_3, P_{014}) = \langle n_{014}, p_3\rangle = \mathcal{V}\,.$$

So the vertices p_2 and p_3 lie on opposite sides of P_{014}, and

$$2\,\mathrm{dist}(p_3, P_{014}) \leq \mathrm{dist}(p_2, P_{014})\,.$$

After relabelling the p_i we proceed again as in Case 1.

A moment's thought gives that the four cases cover all admissible values for $\lambda_1, \lambda_2, \lambda_3$, and so (4.21) holds strictly when no four p_i are coplanar. □

The length quotient for irreducible triply periodic networks of a degree higher than 6 must be larger than the value obtained for the two irreducible networks of degree 5.

Corollary 4.17. *Let N be an irreducible triply periodic network of degree $d \geq 7$. Then*

$$\frac{L^3}{V} > \frac{405}{8}\,. \tag{4.36}$$

Thus for dimension $n = 3$ the length quotient of networks with degree $d \geq 7$ is always larger than the quotient for all explicitly discussed cases with degree 3 to 6.

Proof. For even $d \geq 8$ the quotient network N/Λ covers the bouquet graph $B_{d/2}$. Then (4.36) follows immediately from Theorem 4.10, as

$$\frac{L^3}{V} \geq \left(\frac{8}{2} - 3 + 1\right) 3^3 = 54\,. \tag{4.37}$$

For odd degree d the quotient network N/Λ is classified by Proposition 4.4: It covers the double bouquet graph $D_{\ell,k}$ with $k \geq 3$ and $\ell \geq 0$. Assume first the number of loops in $D_{\ell,k}$ which lift to generators of the lattice Λ is at least 3. Then $\ell \geq 2$ and N/Λ contains a (possibly disconnected) subgraph N'/Λ which consists of four closed geodesics in $\mathbb{R}^3/\Lambda$, three of which lift to generators of Λ. Note that the length of a closed geodesic is invariant under translation. So we may estimate the length of N'/Λ by a network where the four geodesics intersect at one vertex. The reasoning of the proof of Theorem 4.10 then yields (4.37).

Now suppose that the loops in $D_{\ell,k}$ lift to at most two generators of Λ. If exactly two loops lift to generators of Λ, then if necessary we reason as before to assume that each lift is based at a different vertex of N/Λ. Thus in any case N contains a subnetwork $N' \subsetneq N$ which is triply periodic and covers $D_{1,3}$ or D_5. We conclude the length quotient of N' is estimated by Theorem 4.15 or 4.16, and N has a strictly larger quotient, as desired. $\square$

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